Adult Basic Education

Mathematics

Mathematics 3104B

Exponents and Logarithms

Curriculum Guide

Prerequisites: Math 1104A, Math 1104B and Math 1104C
Math 2104A, Math 2104B and Math 2104C
Math 3104A

Credit Value: 1

Required Mathematics Courses [Degree and Technical Profile]
Mathematics 1104A
Mathematics 1104B
Mathematics 1104C
Mathematics 2104A
Mathematics 2104B
Mathematics 2104C
Mathematics 3104A
Mathematics 3104B
Mathematics 3104C

Table of Contents

To the Instructor

I. Introduction to Mathematics 3104B

This course is an introduction to exponential and logarithmic functions, equations and graphs which is an essential component of post-secondary Math courses.

The student will learn to interpret information from equations, graphs and written descriptions and to translate between these different ways of presentation. Other topics involve a variety of strategies for simplifying or solving exponential and logarithmic equations. Real-life situations demonstrating exponential or logarithmic behavior include population growth (human or bacteria), radioactive decay, earthquake and sound intensity and PH levels.

II. Prerequisites

Students taking this course need to be able to evaluate expressions involving integer exponents. They should be able to convert a percent into a decimal, and calculate percent increases or decreases. There are exercises, (with solutions) for each **Section** of the textbook in **Prerequisites** in the *Teacher's Resource Book*. The instructor may use these problems to determine the student's ability. In the Study Guide, students are often directed to see the instructor for **Prerequisite** exercises. The instructor, however, must ensure that the student has mastered these concepts before beginning a particular unit.

III. Textbook

Note: The Study Guide for this course has an Appendix A**,** which contains copies of pages scanned from *Mathematics 10*.

Most of the concepts are introduced, developed and explained in the **Examples**. The instructor must insist that the student carefully studies and understands each **Example** before moving on to the **Exercises**. In the Study Guide, the student is directed to see the instructor if there are any difficulties.

There are four basic categories included in each section of the textbook which require the student to complete questions:

1. **Investigate**
2. **Discussing the Ideas**
3. **Exercises**
4. **Communicating the Ideas**

Investigate: This section looks at the thinking behind new concepts. The answers to its questions are found in the back of the text.

Discussing the Ideas: This section requires the student to write a response which clarifies and demonstrates understanding of the concepts introduced. The answers to these questions are not in the student text but are in the *Teacher's Resource Book*. Therefore, in the Study Guide, the student is directed to see the instructor for correction. This will offer the instructor some perspective on the extent of the student's understanding. If necessary, reinforcement or remedial work can be introduced. Students should not be given the answer key for this section as the opportunity to assess the student's understanding is then lost.

Exercises: This section helps the student reinforce understanding of the concepts introduced. There are three levels of **Exercises**:

A: direct application of concepts introduced
B: multi-step problem solving and some real-life situations
C: problems of a more challenging nature

The answers to the **Exercises** questions are found in the back of the text.

Communicating the Ideas: This section helps confirm the student's understanding of a particular lesson by requiring a clearly written explanation. The answers to **Communicating the Ideas** are not in the student text, but are in the *Teacher's Resource Book*. In the Study Guide students are asked to see the instructor for correction.

IV. Technology

It is important that students have a **scientific** calculator for their individual use. Ensure that the calculators used have the word "scientific" on it as there are calculators designed for calculation in other areas such as business or statistics which would not have the functions needed for study in this area.

A graphing calculator should be **available** to the students since the text provides many opportunities for its use. The *Teacher's Resource Book* suggests many opportunities to use a graphing calculator. These suggestions are outlined where there is the heading *Integrating Technology*. In the Study Guide, students are directed to see the instructor when a graphing calculator is required. The *Teacher's Resource Book* contains a module called **Graphing Calculator Handbook** which will help the instructor and student get acquainted with some of the main features of the TI-83 Plus graphing calculator.

Graphing software such as *Graphmatica* or *Winplot* can also be used if students do not have access to a graphing calculator but do have access to a computer. The textbook does not offer the same guidance for graphing with these tools as it does for a graphing calculator but each software program does have a HELP feature to answer questions.

V. Curriculum Guides

Each new ABE Mathematics course has a Curriculum Guide for the instructor and a Study Guide for the student. The Curriculum Guide includes the specific curriculum outcomes for the course. Suggestions for teaching, learning, and assessment are provided to support student achievement of the outcomes. Each course is divided into units. Each unit comprises a **two-page layout of four columns** as illustrated in the figure below. In some cases the four-column spread continues to the next two-page layout.

Curriculum Guide Organization:
The Two-Page, Four-Column Spread

Unit Number - Unit Title	
Outcomes Specific curriculum outcomes for the unit.	**Notes for Teaching and Learning** Suggested activities, elaboration of outcomes, and background information.

Unit Number - Unit Title	
Suggestions for Assessment Suggestions for assessing students' achievement of outcomes.	**Resources** Authorized and recommended resources that address outcomes.

To the Instructor

VI. Study Guides

The Study Guide provides the student with the name of the text(s) required for the course and specifies the sections and pages that the student will need to refer to in order to complete the required work for the course. It guides the student through the course by assigning relevant reading and providing questions and/or assigning questions from the text or some other resource. Sometimes it also provides important points for students to note. (See the *To the Student* section of the Study Guide for a more detailed explanation of the use of the Study Guides.) The Study Guides are designed to give students some degree of independence in their work. Instructors should note, however, that there is much material in the Curriculum Guides in the *Notes for Teaching and Learning* and *Suggestions for Assessment* columns that is not included in the Study Guide and instructors will need to review this information and decide how to include it.

VII. Resources

Essential Resources

Addison Wesley Mathematics 12 (Western Canadian edition) Textbook
ISBN:0-201-34624-9

Mathematics 12 Teacher's Resource Book (Western Canadian edition)
ISBN: 0-201-34626-5

Math 3104B Study Guide

Recommended Resources

*Mathematics 12 Independent Study Guide (*Western Canadian edition)
ISBN: 0-201-34625-7

Center for Distance Learning and Innovation: http://www.cdli.ca
Winplot: http://math.exeter.edu/rparris/winplot.html
(Free graphing software)
Graphmatica (Evaluation software available on CD-ROM contained in *Teacher's Resource Book*)
CD Rom accompanying *Teacher's Resource Book*
This CD contains selected solutions from the text and self test solutions from the *Independent Study Guide*.

To the Instructor

Other Resources

Math Links: http://mathforum.org

http://spot.pcc.edu/~ssimonds/winplot
(Free videos concerning Winplot)

http://www.pearsoned.ca/school/math/math/

http://www.ed.gov.nl.ca/edu/sp/mathlist.htm
(Math Companion 3204 Supporting Document)

VIII. Recommended Evaluation

Written Notes	10%
Assignments	10%
Test(s)	30%
Final Exam *(entire course)*	50%
	100%

The overall pass mark for the course is 50%.

Exponents and Logarithms

Outcomes

1.1 Solve problems involving rational exponents.

1.1.1 Simplify and evaluate expressions containing integer exponents.

1.1.2 Apply the exponent laws for powers or variables with rational exponents.

Notes for Teaching and Learning

This is a review of the exponent laws which are an important prerequisite to the complete understanding of rational exponents.

Given $\sqrt[n]{x}$, the x under the root symbol is called the *radicand*, and the small n indicating the root is called the *index*.
For example, given $\sqrt[3]{125}$, 125 is the radicand and 3 is the index.

Students should see that the exponent laws learned earlier for integer exponents also apply when the exponent is a rational number (fraction).

Students may need to be reminded that a power with a rational exponent can be written in two different ways. For example :

$$8^{2/3} = (8^{1/3})^2 = (\sqrt[3]{8})^2$$

and

$$8^{2/3} = (8^2)^{1/3} = \sqrt[3]{8^2}$$

Suggestions for Assessment

Study Guide questions 1.1 to 1.4 will meet the objectives of Outcome 1.1.

See the *Teacher's Resource Book*, *Mathematics 10*, pages 4 - 9, for extra problems that can be used for assessment or review.

See the *Independent Study Guide*, *Mathematics 10*, pages 29 - 36 for Key Terms, Comprehension Questions, a Self-Test and Further Practice that can be used for assessment or review.

There are three Worksheets (with answers) in Appendix B that could be used for review.

- Rational Exponents
- Equations with Rational Exponents
- Exponential Equations

Resources

Appendix A, pages 23 - 42

Appendix B, pages 43 - 50

Teacher's Resource Book, *Mathematics 10*, Chapter 2, pages 4 - 9

Independent Study Guide, Mathematics 10, pages 29 - 36

www.cdli.ca
Math 3204, Unit 03, Sections 03 and 04

Unit 2: Exponential Functions

Outcomes

2.1 Use exponential functions to solve problems.

2.1.1 Give a definition of:
- (i) exponential function
- (ii) exponential growth
- (iii) exponential decay

2.1.2 Compare the equations and graphs of exponential growth and exponential decay, and state the similarities and differences.

2.1.3 Use technology to develop a model to describe exponential data.

2.1.4 Analyze and solve problems using the above models.

Notes for Teaching and Learning

In this introduction to exponential functions and graphs, students should note the components of the function and the characteristic shape of the exponential graph.

The textbook gives a variety of applications involving exponential behavior, and asks the student to find specific information by interpreting it from the graph or by calculating it through the function.

Unit 2: Exponential Functions

Suggestions for Assessment

Study Guide questions 2.1 to 2.5 will meet the objectives of Outcome 2.1.

See the *Teacher's Resource Book*, *Mathematics 12*, pages 5 - 8, for extra problems that can be used for assessment or review.

See the *Independent Study Guide*, *Mathematics 12*, pages 29 - 40 for Key Terms, Comprehension Questions, a Self - Test and Further Practice that can be used for assessment or review.

Resources

Mathematics 12
Section 2.1, Introduction to Exponential Functions,
pages 66 - 73

Teacher's Resource Book, *Mathematics 12*, Chapter 2, pages 5 - 8

Independent Study Guide, Mathematics 10, pages 29 - 40

Unit 3: Logarithms and Their Properties

Outcomes

3.1 Describe the relationship between a logarithmic and exponential function.

3.1.1 Define the term *logarithm* $\left(\log_a x\right)$.

3.1.2 State the relationship between an exponential and a logarithm function.

3.1.3 Evaluate logarithms.

3.1.4 Convert from exponential form to logarithmic form and vice versa.

Notes for Teaching And Learning

Hint: In **Visualizing,** on page 76 in *Mathematics 12*, the base and the exponent are identified in both the log and exponential equation. To help students remember their placement - in the exponential form at least - the instructor might suggest that the *b*ase could be referred to as "*b*acon" and the *e*xponent as "*e*ggs" to support the idea that " $bacon^{eggs}$ " go together.

Suggestions for Assessment

Study Guide questions 3.1 to 3.6 will meet the objectives of Outcome 3.1.

See the *Teacher's Resource Book*, *Mathematics 12*, pages 9 - 11, for extra problems that can be used for assessment or review.

See the *Independent Study Guide*, *Mathematics 12*, pages 29 - 40 for Key Terms, Comprehension Questions, a Self - Test and Further Practice that can be used for assessment or review.

Resources

Section 2.2, Defining a Logarithm, pages 74 - 78

Teacher's Resource Book, *Mathematics 12*, Chapter 2, pages 9 - 11

Independent Study Guide, Mathematics 12, pages 29 - 40

Unit 3: Logarithms and Their Properties

Outcomes

3.2 Use the laws of logarithms to simplify expressions and solve exponential equations.

3.2.1 State the law of logarithms for powers.

$$\log_a x^n = n\log_a x$$

3.2.2 Use the law of logarithms for powers to remove a variable from the position of the exponent.

3.2.3 Solve exponential equations by taking the log of both sides of the equation.

3.2.4 State the law of logarithms for multiplication.

$$\log_a (xy) = \log_a x + \log_a y$$

3.2.5 State the law of logarithms for division.

$$\log_a (\frac{x}{y}) = \log_a x - \log_a y$$

3.2.6 Use the laws of logarithms to simplify expressions.

3.2.7 Evaluate log expressions.

Notes for Teaching and Learning

It is not necessary that students be able to do the derivation for the log laws. It would be a worthwhile activity for the instructor to go through the derivation with the student for the purpose of demonstrating how a law was developed and showing some techniques of problem solving.

The law of logarithms for powers will provide a means of solving for the exponent when it is the unknown or the desired variable.

Instructors should reinforce the idea that an equation is a balance between the left and the right sides of the equal sign. Anything can be done to the equation as long as it is done to both sides. The log is introduced on one side as a means of accessing the law of logarithms for powers but it must also be introduced on the other side in order to maintain this balance.

Suggestions for Assessment

Study Guide questions 3.7 to 3.9 will meet the objectives of Outcome 3.2.

See the *Teacher's Resource Book*, *Mathematics 12*, pages 12 to 14, for extra problems that can be used for assessment or review.

See the *Independent Study Guide*, *Mathematics 12*, pages 29 to 40 for Key Terms, Comprehension Questions, a Self - Test and Further Practice that can be used for assessment or review.

Resources

Mathematics 12,
Section 2.3, The Laws of Logarithms, pages 79 - 85

Teacher's Resource Book, *Mathematics 12*, Chapter 2, pages 12 - 14

Independent Study Guide, Mathematics 12, pages 29 - 40

www.cdli.ca
Math 3204, Unit 03, Section 06

Outcomes

4.1 Use exponential functions to solve problems involving half-lives and doubling times.

4.1.1 State the role of A, b and x in the exponential equations $y = Ab^x$ when $b > 1$ and when $0 < b < 1$.

4.1.2 Determine exponential growth and decay equations for specific situations.

4.1.3 Use log laws to solve growth and decay equations.

4.1.4 Use exponential functions to find specific information in growth and decay problems.

Notes for Teaching and Learning

When something is **growing exponentially**, it will become twice as large within a fixed length of time. Most of the exponential growth problems involve functions of the type, $y = Ab^x$
where A = initial amount , x = time, b = the base.
($b > 1$ for exponential growth)
The doubling time depends on the base, b, and not on A.

The doubling equation has the form $y = A(2^{\frac{t}{d}})$ where
A = initial amount , t = elapsed time, d = doubling time

Either or the above forms can be used depending on the information given in the problem.

A similar property applies when something is **decaying exponentially**. It will become one-half as large within a fixed period of time. The equations used can take the form of $y = Ab^x$, ($0 < b < 1$ for exponential decay).

The half-life equation has the form $y = A(0.5)^{\frac{t}{h}}$ where
A = initial amount
t = time elapsed
h = half-life
Instructors should ensure that students understand that the amount of time required for a population to double in a growth situation remains constant. It does not depend on the initial amount but only on the growth factor. Similarly, in a decay situation, the time required for a population to become half its original size is constant. Students may need to be reminded that when solving exponential equations, such as the one in **Example 1** on page 87, the constant (2.28 in this example) should not be multiplied by the base (1.014). The order of operations specifies that the exponent must be evaluated before multiplying.

Unit 4: Real Situations Involving Exponential and Logarithmic Functions

Suggestions for Assessment

Study Guide questions 4.1 to 4.4 will meet the objectives of Outcome 4.1.

See the *Teacher's Resource Book*, *Mathematics 12*, pages 15 to 17, for extra problems that can be used for assessment or review.

See the *Independent Study Guide*, *Mathematics 12*, pages 29 to 40 for Key Terms, Comprehension Questions, a Self - Test and Further Practice that can be used for assessment or review.

Resources

Mathematics 12,
Section 2.4 , Modeling Real Situations Using Exponential Functions: Part 1, pages 86-94

Teacher's Resource Book, *Mathematics 12*, Chapter 2, pages 15 - 17

Independent Study Guide, Mathematics 12, pages 29 - 40

Unit 4: Real Situations Involving Exponential and Logarithmic Functions

Outcomes

4.2 Use exponential equations and logarithms to model real situations.

4.2.1 Define the following term:
i) logarithmic scale.

4.2.2 Create exponential equations from word problems that give varied information.

4.2.3 Using exponential information, find comparative answers.

Notes for Teaching and Learning

Sometimes specific information, such as the initial size of a population, is not known; yet it is still possible to find an answer that is comparative. The answer describes an **increase** or **decrease** in population size over a period of time.

This comparative method is widely used when describing earthquake intensity, sound intensity and alkalinity of solutions.

Note: This section has appropriate applications for other courses: knowledge of the pH scale will be needed for Acids and Bases in Chemistry and the decibel scale will be needed for Sound in Physics.

All exponential equations in this section will have the same type of information though the specific details will vary depending on the problem.
In the case of population, there will be a need for two population references in the formula; the initial population, P_0 , and the population at a later time, P. The constant increase will be the base. There will be two different times; one will be the time for the constant increase and the other will be the elapsed time.
The instructors should work through several examples with the students and point out the significant information which is the first critical step of problem solving.

Unit 4: Real Situations Involving Exponential and Logarithmic Functions

Suggestions for Assessment

Study Guide questions 4.5 and 4.6 will meet the objectives of Outcome 4.2.

See the *Teacher's Resource Book*, *Mathematics 12*, pages 18 to 20, for extra problems that can be used for assessment or review.

See the *Independent Study Guide*, *Mathematics 12*, pages 29 to 40 for Key Terms, Comprehension Questions, a Self - Test and Further Practice that can be used for assessment or review.

Resources

Mathematics 12 ,
Section 2.5, Modeling Real Situations Using Exponential Functions: Part II, pages 95 - 101

Teacher's Resource Book, *Mathematics 12*, Chapter 2, pages 18 - 20

Independent Study Guide, Mathematics 12, pages 29 - 40

Unit 5: Exponential Functions and Their Graphs

Outcomes

5.1 Graph and analyze exponential functions, with and without technology.

5.1.1 Define the following terms:
- i) asymptote
- ii) vertical intercept
- iii) horizontal intercept

5.1.2 Identify the following properties from the graph of an exponential function of the form $f(x) = b^x$:
- i) increasing or decreasing
- ii) vertical intercept
- iii) horizontal intercept
- iv) asymptote
- v) domain and range

Notes for Teaching and Learning

Instructors should encourage students to look for the pattern in the table of values given for the function, $f(x) = 2^x$ on page 104 in *Mathematics 12.*

Translating graphs of functions was taught in previous courses. Students should recognize transformations by inspecting the equation of the function. The three most common transformations with exponential functions are: reflections, vertical stretches and horizontal stretches.

If students have difficulty, instructors should assign **Prerequisite** exercises on page 22 in the *Teacher's Resource Book*. If necessary, Chapter 1 in *Mathematics 12* should be reviewed by the student.

The Law of Exponents for multiplication is verified on the top of page 105 in *Mathematics 12.*

Note: The asymptote for equation of the form $f(x) = b^x$ is described as the x-axis. The equation of the x-axis is $y = 0$.

Suggestions for Assessment

Study Guide questions 5.1 to 5.3 will meet the objectives of Outcome 5.1.

See the *Teacher's Resource Book*, *Mathematics 12*, pages 22 to 25, for extra problems that can be used for assessment or review.

See the *Independent Study Guide*, *Mathematics 12*, pages 29 to 40 for Key Terms, Comprehension Questions, a Self - Test and Further Practice that can be used for assessment or review.

Resources

Mathematics 12,
Section 2.6, Analyzing the Graphs of Exponential Functions, pages 104 - 112

Teacher's Resource Book, *Mathematics 12*, Chapter 2, pages 22 - 25

Independent Study Guide, Mathematics 12, pages 29 - 40

Unit 6: Logarithms and Their Graphs

Outcomes

6.1 Graph and analyze a logarithmic function, with and without technology.

6.1.1 Show that taking the inverse of an exponential function creates a logarithmic function.

6.1.2 Graph an exponential function and its inverse by interchanging points and by reflecting the graph across the line $y = x$.

6.1.3 Graph logarithmic functions using a table of values, technology and transformations.

6.1.4 Identify the following properties from the graph of an logarithmic function of the form $f(x) = \log_b x$ $(b > 0, b \neq 1)$.

- i) increasing or decreasing
- ii) vertical intercept
- iii) horizontal intercept
- iv) asymptote
- v) domain and range

6.1.5 Compare the properties of the graphs of exponential functions and logarithmic functions.

Notes for Teaching and Learning

The instructor should assign **Prerequisite** exercises on page 36 in the *Teacher's Resource Book*, *Mathematics 12*, Chapter 2, if students need review on the properties of inverse functions and how to find the inverse of a function algebraically and graphically.

If the word "inverse" is thought of as reflection, then when looking at the tables of values for an exponential function and its inverse, (a logarithmic function), students should see that the x and y values are interchanged.

When the graphs of an exponential function and a logarithmic function are compared, it will be clear that they are reflections of each other across the line $y = x$.

Unit 6: Logarithms and Their Graphs

Suggestions for Assessment

Study Guide questions 6.1 to 6.6 will meet the objectives of Outcome 6.1.

See the *Teacher's Resource Book*, *Mathematics 12*, pages 36 to 39, for extra problems that can be used for assessment or review.

See the *Independent Study Guide*, *Mathematics 12*, pages 29 to 40 for Key Terms, Comprehension Questions, a Self - Test and Further Practice that can be used for assessment or review.

Resources

Mathematics 12,
Section 2.10, Logarithmic Functions, pages 133 - 140

Teacher's Resource Book, *Mathematics 12*, Chapter 2, pages 36 - 39

Independent Study Guide, Mathematics 12, pages 29 - 40

Unit 7: Exponential and Logarithmic Equations and Identities

Outcomes

7.1 Find exact roots for exponential equations.

7.1.1 Describe the difference between approximate numbers and exact numbers.

7.1.2 Solve exponential equations by expressing (when possible) both sides with the same base.

7.1.3 Solve exponential equations by taking the logarithm of both sides.

Notes for Teaching and Learning

When finding exact answers, no technology will be used. In other words, calculators will not be needed. Instructors should ensure that students understand the format of the answer required in this section.

Exercise 6 of the **Investigate** on page 146 of *Mathematics 12* will show that not all equations can be solved using all 5 approaches.

The **Investigate, Discussing the Ideas** and **Communicating the Ideas** are very useful sections for this unit. A student must be aware of different strategies, and be able to assess an equation to determine which strategy is appropriate or preferable.

Instructors should encourage the students to spend time on these activities and ensure that **Investigate** and **Discussing the Ideas** are corrected before the students move on to **Exercises.** By the time they reach **Communicating the Ideas**, students should be able to answer the questions with knowledge and experience.

Unit 7: Exponential and Logarithmic Equations and Identities

Suggestions for Assessment

Study Guide questions 7.1 to 7.3 will meet the objectives of Outcome 7.1.

See the *Teacher's Resource Book*, *Mathematics 12*, pages 42 to 44, for extra problems that can be used for assessment or review.

See the *Independent Study Guide*, *Mathematics 12*, pages 29 to 40 for Key Terms, Comprehension Questions, a Self - Test and Further Practice that can be used for assessment or review.

Resources

Mathematics 12
Section 2.11, Revisiting Exponential Functions, pages 146 - 149

Teacher's Resource Book, *Mathematics 12*, Chapter 2, pages 42 - 44

Independent Study Guide, Mathematics 12, pages 29 - 40

Outcomes

7.2 Use the definition of a logarithm and the laws of logarithms to solve logarithmic equations and identities.

7.2.1 Define the following:
i) identity
ii) extraneous root

7.2.2 Solve equations which have both sides expressed as logarithms with the same base.

7.2.3 Solve equations which have some terms expressed as logarithms and some as numbers.

7.2.4 Verify that the solutions to logarithmic equations are valid.

7.2.5 Prove identities using laws of logarithms.

Notes for Teaching and Learning

When solving log equations, students should be guided to check the solutions in the *original* equation to ensure that each solution is valid.

In **Example 1a)** on page 150 in the *Mathematics 12*, $x = -2$ is an extraneous root because when this value is substituted into the first term it becomes $\log_5(-1)$, which is undefined. Instructors should ensure that students do not generalize from this example to conclude that all negative values are extraneous.

Supplementary Examples, 1b), on page 45 in *Teacher's Resource Book* is an example of an equation which has a negative value for an answer.

Unit 7: Exponential and Logarithmic Equations and Identities

Suggestions for Assessment

Study Guide questions 7.4 to 7.6 will meet the objectives of Outcome 7.2.

See the *Teacher's Resource Book*, *Mathematics 12*, pages 44 to 47, for extra problems that can be used for assessment or review.

See the *Independent Study Guide*, *Mathematics 12*, pages 29 to 40 for Key Terms, Comprehension Questions, a Self - Test and Further Practice that can be used for assessment or review.

Questions chosen from Masters 2.4 to 2.8 could be used on a unit test.

Resources

Mathematics 12,
Section 2.12, Logarithmic Equations and Identities,
pages 150 -153

Teacher's Resource Book, *Mathematics 12*, Chapter 2,
pages 44 - 47
Masters 2.4 - 2.8

Independent Study Guide, Mathematics 12, pages 29 - 40

Appendix A
Exponents

A geometric sequence involves repeated multiplication by the same number, the common ratio. When the first term and the common ratio of a geometric sequence are equal, the terms of the sequence are powers. For example, the terms of the geometric sequence with first term 2 and common ratio 2 are powers of 2.

Numeral form:	2	4	8	16	32	...
Power form:	2^1	2^2	2^3	2^4	2^5	...

Recall that the definition of a power depends on whether the exponent is a positive integer, zero, or a negative integer.

Positive Integral Exponent

$a^n = a \bullet a \bullet a \bullet \cdots \bullet a$

n factors

Zero Exponent

a^0 is defined to be equal to 1.

$$a^0 = 1, \quad (a \neq 0)$$

Negative Integral Exponent

a^{-n} is defined to be the reciprocal of a^n.

$$a^{-n} = \frac{1}{a^n}, \quad (a \neq 0)$$

These definitions permit us to extend the sequence of powers of 2 above to the left.

Multiplying each term by 2 increases the exponent by 1.

$\rightarrow$

...	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	...
...	2^{-4}	2^{-3}	2^{-2}	2^{-1}	2^0	2^1	2^2	2^3	2^4	...

$\leftarrow$

Dividing each term by 2 decreases the exponent by 1.

We use these definitions to evaluate a power with any integral exponent.

Example 1

Simplify each power.

a) 4^3 b) 3^{-2} c) $(-2)^{-3}$ d) $\left(\frac{1}{2}\right)^0$

Solution

Use mental math.

a) $4^3 = 4 \times 4 \times 4$
$= 64$

b) $3^{-2} = \frac{1}{3^2}$
$= \frac{1}{9}$

c) $(-2)^{-3} = \frac{1}{(-2)^3}$
$= -\frac{1}{8}$

d) $\left(\frac{1}{2}\right)^0 = 1$

The definitions of integral exponents lead to some basic laws for working with exponents. These examples will help you recall the laws they illustrate.

Expression	Exponent Law
$x^3 \cdot x^2 = x \cdot x \cdot x \cdot x \cdot x$ $= x^{3+2}$, or x^5	$x^m \cdot x^n = x^{m+n}$
$x^5 \div x^3 = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x}$ $= x^{5-3}$, or x^2	$x^m \div x^n = x^{m-n}$ $(x \neq 0)$
$(x^3)^2 = (x \cdot x \cdot x)(x \cdot x \cdot x)$ $= x^{3 \times 2}$, or x^6	$(x^m)^n = x^{mn}$
$(xy)^3 = xy \cdot xy \cdot xy$ $= x \cdot x \cdot x \cdot y \cdot y \cdot y$ $= x^3y^3$	$(xy)^n = x^ny^n$
$\left(\frac{x}{y}\right)^2 = \frac{x}{y} \cdot \frac{x}{y}$ $= \frac{x^2}{y^2}$	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$ $(y \neq 0)$

We use the exponent laws to simplify products and quotients involving powers.

Example 2

Simplify.

a) $(x^3y^2)(x^2y^4)$ **b)** $\frac{a^5b^3}{a^2b^2}$ **c)** $\left(\frac{x^2}{z^3}\right)^2$

Solution

Use mental math.

a) $(x^3y^2)(x^2y^4)$
$= x^3 \bullet y^2 \bullet x^2 \bullet y^4$
$= x^5y^6$

b) $\frac{a^5b^3}{a^2b^2} = \frac{a^5}{a^2} \bullet \frac{b^3}{b^2}$
$= a^3b$

c) $\left(\frac{x^2}{z^3}\right)^2 = \frac{x^2}{z^3} \bullet \frac{x^2}{z^3}$
$= \frac{x^4}{z^6}$

Example 3

Simplify.

a) $x^{-3} \bullet x^5$ **b)** $m^2 \div m^{-3}$ **c)** $(n^{-2})^{-3}$

Solution

Use mental math.

a) $x^{-3} \bullet x^5 = x^{-3+5}$
$= x^2$

b) $m^2 \div m^{-3} = m^{2-(-3)}$
$= m^5$

c) $(n^{-2})^{-3} = n^{(-2)\times(-3)}$
$= n^6$

Example 4

The number of insects in a colony doubles every month. There are 1000 insects in the colony now. About how many were there in the colony three months ago?

Solution

Let x represent the number of insects 3 months ago. Then, after 3 successive doublings the colony grows to 1000 insects.

$$x \times 2^3 = 1000$$
$$x = \frac{1000}{2^3}$$
$$= 125$$

There were about 125 insects in the colony 3 months ago.

Exercises

1. Simplify.

a) 2^4 b) 5^{-2} c) 3^{-1} d) $\left(\frac{1}{4}\right)^{-1}$

e) $\left(\frac{2}{3}\right)^{-1}$ f) $\left(\frac{3}{4}\right)^{-2}$ g) 0.5^{-1} h) 1.5^0

2. Simplify.

a) 10^0 b) $(-3)^{-2}$ c) $\left(-\frac{1}{2}\right)^3$ d) $\left(-\frac{2}{3}\right)^{-1}$

e) $\left(-\frac{3}{5}\right)^{-2}$ f) $(-1)^{-4}$ g) 0.1^{-4} h) $\frac{1}{2^{-3}}$

3. Choose one part of exercise 1 or 2. Write to explain how you simplified the expression.

4. Simplify.

a) $x^3 \bullet x^4$ b) $a^2 \bullet a^5$ c) $b^3 \bullet b^5 \bullet b$ d) $m^2 \bullet m^3 \bullet m^4$

5. Simplify.

a) $\frac{x^4}{x^2}$ b) $\frac{y^7}{y^3}$ c) $\frac{n^6}{n^5}$ d) $\frac{a^8}{a^5}$

6. Simplify.

a) $(x^3)^3$ b) $(y^2)^3$ c) $(a^2b^2)^3$ d) $(xy^3)^2$

7. Simplify.

a) $\frac{x^2}{x^5}$ b) $\frac{c^3}{c^4}$ c) $\frac{y^2}{y^7}$ d) $\frac{a^2}{a^6}$

8. Choose one part of exercises 4 to 7. Write to explain how you simplified the expression.

9. A colony of 10 000 insects doubles in number every month. How many insects were there at each time?

a) 2 months ago **b)** 5 months ago

10. Simplify.

a) $x^{-3} \bullet x^4$ **b)** $d^{-4} \bullet d^{-1}$ **c)** $a^6 \bullet a^{-2}$ **d)** $y^4 \bullet y^{-4}$

e) $x^{-5} \bullet x^{-1} \bullet x^{-3}$ **f)** $b^4 \bullet b^{-3} \bullet b^2$ **g)** $k^8 \bullet k^{-2} \bullet k^{-6}$ **h)** $p^{-1} \bullet p^7 \bullet p^{-6}$

11. Simplify.

a) $\frac{x^{-5}}{x^2}$ **b)** $\frac{r^3}{r^{-2}}$ **c)** $\frac{s^5}{s^{-5}}$ **d)** $\frac{t^{-4}}{t^{-4}}$

e) $\frac{c^{-1}}{c^{-2}}$ **f)** $\frac{x^{-2}}{x^{-4}}$ **g)** $\frac{b^{-8}}{b^{-3}}$ **h)** $\frac{t^4}{t^{-7}}$

12. Simplify.

a) $(x^{-2})^3$ **b)** $(y^{-1})^{-2}$ **c)** $(m^{-3})^2$ **d)** $(c^3)^{-3}$

e) $(a^4)^{-1}$ **f)** $(x^{-1}y^2)^{-1}$ **g)** $(x^2y^{-3})^2$ **h)** $(a^{-2}b^2)^{-2}$

13. Simplify.

a) $(a^{-2}b^4)(a^2b^{-5})$ **b)** $\frac{(x^{-2})^3}{(x^3)^{-2}}$ **c)** $\frac{x^2y^{-2}}{y^{-1}}$

d) $(x^{-1}y^2)^{-3}(x^2)^{-1}$ **e)** $\frac{(c^{-3}d)^{-1}}{(c^2d)^{-2}}$ **f)** $(m^2n^{-2})(m^{-1}n^2)^{-1}$

14. **a)** Evaluate each expression for $x = -1$ and $y = 2$.

i) $(x^3y^2)(x^2y^3)$ **ii)** $\frac{x^{-4}y^5}{xy^3}$ **iii)** $(x^3y^2)^3$

iv) $(x^{-1}y^{-2})(x^{-2}y^{-3})$ **v)** $\frac{x^{-3}y^{-2}}{x^2y^{-6}}$ **vi)** $(x^{-4}y^{-3})^{-2}$

b) Choose one expression from part a. Write to explain how you evaluated it.

Visualizing

Gold leaf is so thin that one dollar's worth would cover a square with an approximate area of 3600 cm^2.

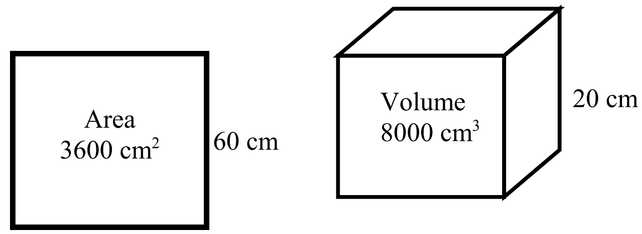
Since $60^2 = 3600$, each side of the square is 60 cm long.

We say that 60 is a square root of 3600, and we write $\sqrt{3600} = 60$.

The gold produced in Canada in one day would almost fill a cube with an approximate volume of 8000 cm^3.

Since $20^3 = 8000$, each edge of the cube is 20 cm long.

We say that 20 is a cube root of 8000, and we write $\sqrt[3]{8000} = 20$.



Expressions such as $\sqrt{3600}$ and $\sqrt[3]{8000}$ are *radicals*. We use radicals when we work with square roots and cube roots of numbers.

Square Roots

A number r is a *square root* of a number x if $r^2 = x$.

A positive number always has two square roots, one positive, the other negative. Since the squares of both positive and negative numbers are positive, it is impossible to obtain a negative number when a number is squared. Hence, the square root of a negative number is not defined.

The *radical sign*, $\sqrt{}$, always denotes the positive square root.

$\sqrt{x}$ **means the positive square root of x, where $x \geq 0$.**

Example 1

Determine each square root, without using a calculator.

a) $\sqrt{1600}$ b) $\sqrt{2.25}$ c) $\sqrt{0.09}$

Solution

a) $\sqrt{1600} = 40$, since $40^2 = 1600$

b) $\sqrt{2.25} = 1.5$, since $1.5^2 = 2.25$

c) $\sqrt{0.09} = 0.3$, since $0.3^2 = 0.09$

In *Example 1*, the square roots were exact. Many numbers do not have exact square roots, but you can use a calculator to determine approximations of them. Since the calculator displays only a fixed number of digits, the number displayed may only be an approximation of the square root. For example, if you use your calculator to determine $\sqrt{7}$, you will obtain 2.645 751 311 (assuming your calculator has 10-digit accuracy). You can obtain different approximations of $\sqrt{7}$ by rounding or by truncating.

Example 2

Sharon used her calculator to determine $\sqrt{7}$. She obtained 2.645 751 311. Write approximations to 1, 2, and 3 decimal places that are obtained by rounding and by truncating. Use your calculator to check your approximations.

Solution

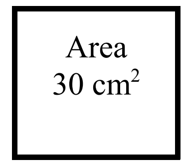
Use a table to record the results.

Number of decimal places	Rounding	Check	Truncating	Check
1	$\sqrt{7} \doteq 2.6$	$2.6 \times 2.6 = 6.76$	$\sqrt{7} \doteq 2.6$	$2.6 \times 2.6 = 6.76$
2	$\sqrt{7} \doteq 2.65$	$2.65 \times 2.65 = 7.0225$	$\sqrt{7} \doteq 2.64$	$2.64 \times 2.64 = 6.9696$
3	$\sqrt{7} \doteq 2.646$	$2.646 \times 2.646 = 7.001\ 316$	$\sqrt{7} \doteq 2.645$	$2.645 \times 2.645 = 6.996\ 025$

In *Example 2*, observe that if you include more decimal places when you estimate the square root, you get closer to 7 when you square the estimate. Furthermore, rounding and truncating provide either the same estimate, or one that differs only by 1 in the final digit.

Example 3

A square has an area of 30 cm². Determine the perimeter of the square in exact form, and in approximate form to 2 decimal places.



Solution

Since $\sqrt{30} \times \sqrt{30} = 30$, the length of each side is $\sqrt{30}$ cm.
Hence, the perimeter is $4 \times \sqrt{30}$ cm, which we write as $4\sqrt{30}$ cm.
In approximate form:

$$4\sqrt{30} \doteq 4 \times 5.4772$$
$$\doteq 21.9089$$

To 2 decimal places, the perimeter of the square is 21.91 cm.

Cube Roots

A number r is a *cube root* of a number x if $r^3 = x$.

The cube root of a positive number is positive and the cube root of a negative number is negative.

$\sqrt[3]{x}$ means the cube root of x.

Example 4

Determine each cube root.

a) $\sqrt[3]{125}$ **b)** $\sqrt[3]{-64}$ **c)** $\sqrt[3]{18}$

Solution

a) Use mental math.
$\sqrt[3]{125} = 5$,
since $5^3 = 125$

b) Use mental math.
$\sqrt[3]{-64} = -4$,
since $(-4)^3 = -64$

c) Use the cube-root function on your calculator. Consult your manual if necessary.
$\sqrt[3]{18} \doteq 2.621$ rounded to 3 decimal places

Higher Roots

Similarly, the *fourth roots* of 16 are 2 and -2, since $2^4 = 16$, and $(-2)^4 = 16$.

We write $\sqrt[4]{16} = 2$ to indicate the positive fourth root of 16.

And the *fifth root* of -32 is -2, since $(-2)^5 = -32$.

We write $\sqrt[5]{-32} = -2$.

An expression of the form $\sqrt[n]{x}$, where n is a natural number, is a *radical*.
If n is even, the expression represents only the positive root.

Exercises

1. Determine the square roots of each number.

a) 49 **b)** 81 **c)** 121 **d)** 400 **e)** 529 **f)** 625

2. Simplify without using a calculator.

a) $\sqrt{64}$ **b)** $\sqrt{100}$ **c)** $\sqrt{144}$ **d)** $\sqrt{900}$ **e)** $\sqrt{1600}$

f) $\sqrt{0.25}$ **g)** $\sqrt{0.04}$ **h)** $\sqrt{0.01}$ **i)** $\sqrt{0.0016}$ **j)** $\sqrt{0.000\ 025}$

3. Use a calculator to determine each square root, to 3 decimal places.

a) $\sqrt{2}$ **b)** $\sqrt{3}$ **c)** $\sqrt{52.3}$ **d)** $\sqrt{128.5}$ **e)** $\sqrt{471}$

4. Simplify without using a calculator.

a) $\sqrt[3]{8}$ **b)** $\sqrt[3]{-27}$ **c)** $\sqrt[4]{81}$

d) $\sqrt[5]{32}$ **e)** $\sqrt[5]{243}$ **f)** $\sqrt[3]{0.001}$

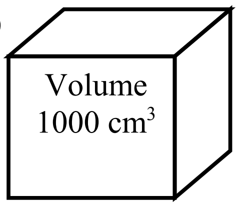
5. Use the area of each square. Determine the length of a side and the perimeter. Give the answer to 1 decimal place, where necessary.

a) Area $9.61\ m^2$

b) Area $6.4\ m^2$

c) Area $8.5\ m^2$

6. Use the volume of each cube. Determine the length of an edge and the area of a face. Give the
answer to 1 decimal place, where necessary.

a)


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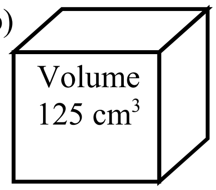
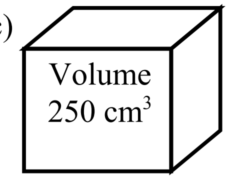


c)


7. Use your calculator. Determine each root to 2 decimal places. Use your calculator to check your approximations.

a) $\sqrt{6}$ b) $\sqrt{11}$ c) $\sqrt[3]{23}$ d) $\sqrt{124}$ e) $\sqrt[3]{139}$ f) $\sqrt[3]{254}$

8. Write each number in square root form.

a) 5 b) 3 c) 2 d) 4 e) 7 f) 1

9. a) Use your calculator to determine $\sqrt{60}$. Write approximations to 1, 2, and 3 decimal places that are obtained by rounding and by truncating. Use your calculator to check your approximations.

b) Suppose you have two estimates of the square root of a number, one obtained by rounding to a certain number of decimal places and the other by truncating to the same number of decimal places. How can you tell which was obtained by rounding and which by truncating? Use your results from part a to support your answer.

c) Suppose you have only one estimate of the square root of a number. Can you tell if it was obtained by rounding or by truncating? Explain.

10. Simplify the radicals in each list without using a calculator. What patterns can you find in the results? Predict the next line in each list.

a) $\sqrt{9}$
$\sqrt{900}$
$\sqrt{90\,000}$

b) $\sqrt[3]{8}$
$\sqrt[3]{8000}$
$\sqrt[3]{8\,000\,000}$

c) $\sqrt[4]{16}$
$\sqrt[4]{160\,000}$
$\sqrt[4]{1\,600\,000\,000}$

11. Simplify without using a calculator.

a) $\sqrt[3]{64}$ b) $\sqrt[3]{125}$ c) $\sqrt[4]{16}$ d) $\sqrt[5]{-1}$ e) $\sqrt[3]{216}$
f) $\sqrt[3]{-1000}$ g) $\sqrt[4]{256}$ h) $\sqrt[4]{10\,000}$ i) $\sqrt[3]{7^3}$ j) $\sqrt[5]{10^5}$

A power with a natural number exponent is defined using repeated multiplication; for example, 3^4 means $3 \times 3 \times 3 \times 3$.

A power with a rational exponent, such as $3^{\frac{1}{2}}$ or $3^{0.5}$, has no meaning according to this definition. It has been defined in another way. You can use your calculator to discover the definition.

The keystrokes are for the TI-34 calculator. If you use a different calculator, consult your manual for the keystrokes.

1. a) To determine $3^{\frac{1}{2}}$, press 3 [y^x] .5 [=]. Record the result.

b) Can you tell how $3^{\frac{1}{2}}$ is defined? If so, use your calculator to confirm this.

2. a) To determine $3^{\frac{1}{3}}$, press 3 [y^x] [(] 1 [÷] 3 [)] [=]. Record the result.

b) Can you tell how $3^{\frac{1}{3}}$ is defined? If so, use your calculator to confirm this.

3. Copy and complete the tables. Use the results. How do you think $x^{\frac{1}{2}}$ and $x^{\frac{1}{3}}$ are defined?

4. Use the results of exercise 3. How would you define $x^{\frac{1}{n}}$?

x	$x^{\frac{1}{2}}$
1	
2	
3	
4	
9	
16	
25	

x	$x^{\frac{1}{3}}$
1	
2	
3	
8	
27	
64	
125	

5. Predict how $x^{-\frac{1}{2}}$ is defined. Copy the first table. Use your calculator to complete it. Did the results agree with your prediction?

6. Copy the second table. Use your calculator to complete it. Use the results. How do you think $x^{\frac{2}{3}}$ is defined?

x	$x^{-\frac{1}{2}}$
1	
2	
3	
4	
9	
16	
25	

x	$x^{\frac{2}{3}}$
1	
2	
3	
8	
27	
64	
125	

7. Use the results of exercises 5 and 6. How would you define $x^{-\frac{1}{n}}$ and $x^{\frac{m}{n}}$?

To give meaning to powers such as $3^{\frac{1}{2}}$ and $3^{-\frac{1}{2}}$, we *extend* the exponent law $x^m \times x^n = x^{m+n}$ so that it applies when m and n are rational numbers.

By extending the law:

$$3^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 3^{\frac{1}{2}+\frac{1}{2}} = 3^1 = 3$$

But: $\sqrt{3} \times \sqrt{3} = 3$

Therefore, $3^{\frac{1}{2}} = \sqrt{3}$

By extending the law:

$$3^{-\frac{1}{2}} \times 3^{\frac{1}{2}} = 3^{-\frac{1}{2}+\frac{1}{2}} = 3^0 = 1$$

Therefore, $3^{-\frac{1}{2}}$ and $3^{\frac{1}{2}}$ are reciprocals.

$$3^{-\frac{1}{2}} = \frac{1}{3^{\frac{1}{2}}} = \frac{1}{\sqrt{3}}$$

These examples and the results of *Investigate* suggest that $x^{\frac{1}{n}}$ should be defined as the nth root of x, and $x^{-\frac{1}{n}}$ as its reciprocal.

$x^{\frac{1}{n}} = \sqrt[n]{x}$ $\quad$ n is a natural number, $x \geq 0$ if n is even.

$x^{-\frac{1}{n}} = \frac{1}{\sqrt[n]{x}}$ $\quad$ n is a natural number, $x \neq 0$, $x > 0$ if n is even.

Example 1

Determine each exact value without using a calculator.

a) $27^{\frac{1}{3}}$

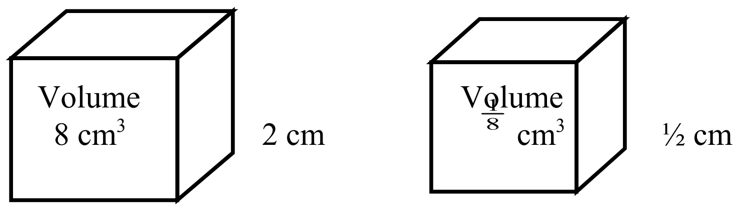
b) $\left(\frac{9}{16}\right)^{-\frac{1}{2}}$

Solution

a) $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

b) $\left(\frac{9}{16}\right)^{-\frac{1}{2}} = \frac{1}{\left(\frac{9}{16}\right)^{\frac{1}{2}}} = \frac{1}{\sqrt{\frac{9}{16}}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$

Visualizing



$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

$8^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$

To give meaning to a power such as $3^{\frac{2}{3}}$ we extend the exponent law $(x^m)^n = x^{mn}$ so that it applies when m and n are rational numbers.

By extending the law:

$$3^{\frac{2}{3}} = (3^{\frac{1}{3}})^2 = (\sqrt[3]{3})^2 \quad \text{or} \quad 3^{\frac{2}{3}} = (3^2)^{\frac{1}{3}} = \sqrt[3]{3^2}$$

These examples and the results of *Investigate* suggest the following definitions for $x^{\frac{m}{n}}$.

$$x^{\frac{m}{n}} = (\sqrt[n]{x})^m = \sqrt[n]{x^m}$$

n is a natural number, $x \geq 0$ if n is even.

To give meaning to a power such as $3^{-\frac{2}{3}}$, we use the law $x^m \bullet x^n = x^{m+n}$, which we extended to rational exponents previously.

$$3^{-\frac{2}{3}} \times 3^{\frac{2}{3}} = 3^{-\frac{2}{3}+\frac{2}{3}} = 3^0 = 1$$

Therefore, $3^{-\frac{2}{3}}$ and $3^{\frac{2}{3}}$ are reciprocals.

$$3^{-\frac{2}{3}} = \frac{1}{3^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{3})^2} \quad \text{or} \quad \frac{1}{\sqrt[3]{3^2}}$$

This example suggests the following definitions for $x^{-\frac{m}{n}}$.

$$x^{-\frac{m}{n}} = \frac{1}{(\sqrt[n]{x})^m}$$
$$= \frac{1}{\sqrt[n]{x^m}}$$

n is a natural number, $x \neq 0$, $x > 0$ if n is even.

Example 2

Determine each exact value without using a calculator.

a) $27^{\frac{2}{3}}$ b) $\left(\frac{9}{16}\right)^{-\frac{3}{2}}$

Solution

a) $27^{\frac{2}{3}} = (\sqrt[3]{27})^2$
$= 3^2$
$= 9$

b) $\left(\frac{9}{16}\right)^{-\frac{3}{2}} = \frac{1}{\left(\frac{9}{16}\right)^{\frac{3}{2}}}$
$= \frac{1}{\left(\sqrt{\frac{9}{16}}\right)^3}$
$= \frac{1}{\left(\frac{3}{4}\right)^3}$
$= \frac{1}{\frac{27}{64}}$
$= \frac{64}{27}$

Visualizing

$8^{\frac{2}{3}} = (\sqrt[3]{8})^2$ ← means square of the cube root
$= 2^2$
$= 4$

$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}}$ ← means reciprocal
$= \frac{1}{(\sqrt[3]{8})^2}$
$= \frac{1}{2^2}$
$= \frac{1}{4}$

You can use the laws of exponents to simplify expressions involving radicals and rational exponents.

Example 3

Simplify. Write each expression as a power and as a radical.

a) $\sqrt[6]{x^3}$ b) $\sqrt{\sqrt[3]{x^5}}$ c) $(\sqrt[3]{x^4})(\sqrt{x^3})$

Solution

a) $\sqrt[6]{x^3} = (x^3)^{\frac{1}{6}}$ Using the law: $\sqrt[n]{x^m} = (x^m)^{\frac{1}{n}}$

$= x^{3 \times \frac{1}{6}}$ Using the law: $(x^m)^n = x^{mn}$

$= x^{\frac{1}{2}}$, or $\sqrt{x}$

b) $\sqrt{\sqrt[3]{x^5}} = (\sqrt[3]{x^5})^{\frac{1}{2}}$ Using the law: $\sqrt[n]{x^m} = (x^m)^{\frac{1}{n}}$

$= ((x^5)^{\frac{1}{3}})^{\frac{1}{2}}$ Using the law: $\sqrt[n]{x^m} = (x^m)^{\frac{1}{n}}$ again

$= x^{5 \times \frac{1}{3} \times \frac{1}{2}}$

$= x^{\frac{5}{6}}$, or $\sqrt[6]{x^5}$

c) $(\sqrt[3]{x^4})(\sqrt{x^3}) = (x^4)^{\frac{1}{3}} \times (x^3)^{\frac{1}{2}}$ Using the law: $\sqrt[n]{x^m} = (x^m)^{\frac{1}{n}}$ twice

$= x^{\frac{4}{3}} \times x^{\frac{3}{2}}$ Using the law: $(x^m)^n = x^{mn}$

$= x^{\frac{8}{6} + \frac{9}{6}}$ Using the law: $x^m \times x^n = x^{m+n}$

$= x^{\frac{17}{6}}$, or $(\sqrt[6]{x})^{17}$

DISCUSSING THE IDEAS

1. Think about the definition of a rational exponent. Why is there a restriction on x if n is even? Use some examples to illustrate your explanation.

2. Determine another way to solve *Example 1b* and *Example 2b* that involves fewer steps.

Exercises

1. Determine each exact value without using a calculator.

a) 8^0 b) $8^{\frac{1}{3}}$ c) $8^{\frac{2}{3}}$ d) $8^{\frac{3}{3}}$ e) $8^{\frac{4}{3}}$

f) $8^{-\frac{1}{3}}$ g) $8^{-\frac{2}{3}}$ h) $8^{-\frac{3}{3}}$ i) $8^{-\frac{4}{3}}$ j) $8^{-\frac{5}{3}}$

2. Determine each exact value without using a calculator.

a) $16^{\frac{1}{2}}$ b) $36^{\frac{1}{2}}$ c) $100^{\frac{1}{2}}$ d) $32^{\frac{1}{5}}$ e) $64^{\frac{1}{3}}$

f) $27^{\frac{1}{3}}$ g) $(-64)^{\frac{1}{3}}$ h) $81^{\frac{1}{4}}$ i) $(-27)^{\frac{1}{3}}$ j) $(-1000)^{\frac{1}{3}}$

3. Determine each exact value without using a calculator.

a) $4^{-\frac{1}{2}}$ b) $9^{-\frac{1}{2}}$ c) $27^{-\frac{1}{3}}$ d) $64^{-\frac{1}{3}}$ e) $(-64)^{-\frac{1}{3}}$

4. Write each expression using radicals.

a) $4^{\frac{1}{5}}$ b) $4^{\frac{2}{5}}$ c) $4^{\frac{3}{5}}$ d) $4^{\frac{4}{5}}$ e) $4^{\frac{5}{5}}$

f) $4^{-\frac{1}{5}}$ g) $4^{-\frac{2}{5}}$ h) $4^{-\frac{3}{5}}$ i) $4^{-\frac{4}{5}}$ j) $4^{-\frac{5}{5}}$

5. Choose one part of exercise 4. Write to explain how you wrote the power as a radical.

6. Determine each exact value without using a calculator.

a) $9^{\frac{3}{2}}$ b) $27^{\frac{2}{3}}$ c) $4^{\frac{3}{2}}$ d) $25^{\frac{3}{2}}$ e) $32^{\frac{2}{5}}$

f) $(-27)^{\frac{2}{3}}$ g) $36^{\frac{3}{2}}$ h) $(-64)^{\frac{2}{3}}$ i) $100^{\frac{3}{2}}$ j) $(-8000)^{\frac{2}{3}}$

7. Determine each exact value without using a calculator.

a) $27^{-\frac{2}{3}}$ b) $32^{-\frac{3}{5}}$ c) $9^{-\frac{3}{2}}$ d) $16^{-\frac{3}{4}}$ e) $100^{-\frac{3}{2}}$

8. Write each number as a power with an exponent of $\frac{1}{2}$.

a) 3 b) 2 c) 4 d) 1 e) 10 f) 8

9. Choose one part of exercise 8. Write to explain how you wrote the number as a power.

10. Determine each exact value without using a calculator.

a) $27^{-\frac{4}{3}}$ b) $16^{-1.5}$ c) $81^{0.75}$ d) $32^{-0.4}$ e) $49^{\frac{3}{2}}$

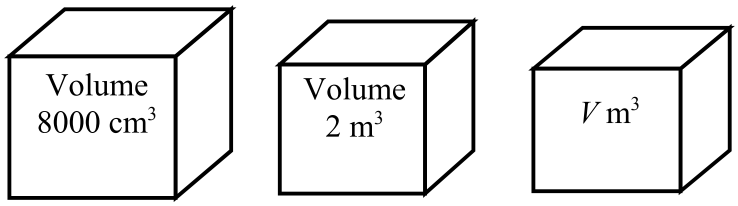
f) $\left(\frac{9}{16}\right)^{\frac{1}{2}}$ g) $\left(\frac{25}{49}\right)^{\frac{3}{2}}$ h) $\left(-\frac{1}{32}\right)^{0.8}$ i) $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$ j) $\left(\frac{81}{16}\right)^{-\frac{3}{4}}$

11. Choose one part of exercise 10. Write to explain how you determined the exact value.

12. Use a calculator. Determine each value to 3 decimal places.

a) $10^{\frac{1}{4}}$ b) $30^{0.7}$ c) $7^{\frac{2}{3}}$ d) $15^{1.4}$ e) $\sqrt[8]{2.17}$

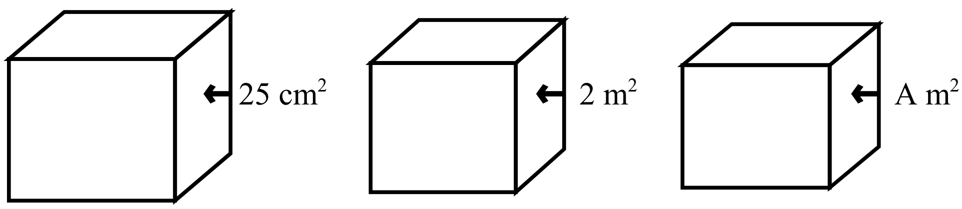
13. a) A cube has a volume of 8000 cm^3. Determine the length of each edge and the area of each face.



b) Repeat part a for a cube with a volume of 2 m^3. Express your answers in radical form.

c) Repeat part a for a cube with a volume of V cubic metres.

14. a) The area of each face of a cube is 25 cm^2. Determine the length of each edge and the volume of the cube.



b) Repeat part a for a cube with each face with area 2 m^2. Express your answers in radical form.

c) Repeat part a for a cube with each face with area A square metres.

15. Write each number as a power with an exponent of $\frac{1}{3}$.

a) 3 **b)** −1 **c)** −2 **d)** −4 **e)** 1 **f)** −3

16. Simplify. Write each expression as a power and as a radical.

a) $x \bullet x^{\frac{1}{2}}$ **b)** $m^{\frac{1}{3}} \bullet m$ **c)** $y^{\frac{3}{2}} \bullet y^{\frac{1}{2}}$ **d)** $b^{\frac{5}{2}} \bullet b^{\frac{3}{2}}$

e) $x \div x^{\frac{1}{2}}$ **f)** $m \div m^{\frac{1}{3}}$ **g)** $d^{\frac{3}{2}} \div d^{\frac{1}{2}}$ **h)** $p^{\frac{3}{5}} \div p^{\frac{1}{5}}$

Appendix B
Worksheets

Rational Exponents

Write in radical form.

1. $2^{\frac{1}{3}}$ 2. $37^{\frac{3}{2}}$ 3. $x^{\frac{1}{2}}$ 4. $a^{\frac{1}{5}}$ 5. $6^{\frac{4}{3}}$ 6. $6^{\frac{3}{4}}$

7. $7^{-\frac{1}{2}}$ 8. $9^{-\frac{1}{5}}$ 9. $x^{-\frac{3}{7}}$ 10. $b^{-\frac{6}{5}}$ 11. $(3x)^{\frac{1}{2}}$ 12. $3x^{\frac{1}{2}}$

Write using exponents.

13. $\sqrt{7}$ 14. $\sqrt{34}$ 15. $\sqrt[3]{-11}$ 16. $\sqrt[5]{a^2}$ 17. $\sqrt[3]{6^4}$ 18. $(\sqrt[3]{b})^4$

19. $\frac{1}{\sqrt{x}}$ 20. $\frac{1}{\sqrt[3]{a}}$ 21. $\frac{1}{(\sqrt[5]{x})^4}$ 22. $\sqrt[3]{2b^3}$ 23. $\sqrt{3x^5}$ 24. $\sqrt[4]{5t^3}$

Evaluate without using a calculator..

25. $4^{\frac{1}{2}}$ 26. $125^{\frac{1}{3}}$ 27. $16^{-\frac{1}{4}}$ 28. $(-32)^{\frac{1}{5}}$ 29. $25^{0.5}$ 30. $(-27)^{-\frac{1}{3}}$

31. $(64)^{-\frac{1}{6}}$ 32. $0.04^{\frac{1}{2}}$ 33. $81^{0.25}$ 34. $0.001^{\frac{1}{3}}$ 35. $\left(\frac{4}{9}\right)^{\frac{1}{2}}$ 36. $\left(\frac{-27}{-8}\right)^{\frac{1}{3}}$

37. $8^{\frac{2}{3}}$ 38. $4^{\frac{3}{2}}$ 39. $9^{2.5}$ 40. $81^{\frac{3}{4}}$ 41. $16^{-\frac{3}{4}}$ 42. $(-32)^{\frac{2}{5}}$

43. $(-8)^{-\frac{5}{3}}$ 44. $(-27)^{-\frac{2}{3}}$ 45. $1^{\frac{5}{3}}$ 46. $(-1)^{-\frac{8}{5}}$ 47. $\left(\frac{100}{9}\right)^{\frac{3}{2}}$ 48. $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$

Simplify without using a calculator.

49. a) $9^{\frac{1}{2}}$ b) $9^{-\frac{1}{2}}$ 50. a) $8^{\frac{1}{3}}$ b) $8^{-\frac{1}{3}}$

51. a) $8^{\frac{2}{3}}$ b) $8^{-\frac{2}{3}}$ 52. a) $16^{\frac{1}{4}}$ b) $16^{-\frac{1}{4}}$

53. $27^{-\frac{1}{3}}$ 54. $27^{\frac{2}{3}}$ 55. $4^{\frac{3}{2}}$ 56. $25^{-\frac{1}{2}}$

57. $81^{\frac{1}{2}}$ 58. $49^{-\frac{1}{2}}$ 59. $4^{\frac{3}{2}}$ 60. $16^{\frac{3}{4}}$

61. $(-125)^{-\frac{1}{3}}$ 62. $4^{-0.5}$ 63. $-8^{\frac{2}{3}}$ 64. $(5^{\frac{1}{3}})^{-3}$

65. $(16^{-5})^{\frac{1}{20}}$ 66. $(9^{\frac{1}{2}}+16^{\frac{1}{2}})^{2}$

Equations with Rational Exponents

Give the power to which you would raise both sides of each equation in order to solve the equation.

1. $x^{\frac{1}{2}} = 9$ 2. $x^{\frac{2}{3}} = 4$ 3. $x^{-\frac{1}{3}} = 2$ 4. $x^{-\frac{3}{4}} = 8^{-1}$

Solve each equation.

5. a) $a^{\frac{3}{4}} = 8$ b) $(3x+1)^{\frac{3}{4}} = 8$

6. a) $y^{-\frac{1}{2}} = 6$ b) $(3y)^{-\frac{1}{2}} = 6$

7. a) $2y^{-\frac{1}{2}} = 10$ b) $(2y)^{-\frac{1}{2}} = 10$

8. a) $(9t)^{-\frac{2}{3}} = 4$ b) $9t^{-\frac{2}{3}} = 4$

9. $(8-y)^{\frac{1}{3}} = 4$ 10. $(3n-1)^{-\frac{2}{3}} = \frac{1}{4}$

11. $(x^2+4)^{\frac{2}{3}} = 25$ 12. $(x^2+9)^{\frac{1}{2}} = 5$

Exponential Equations

Solve each of the following equations.

1. $3^x = \frac{1}{9}$
2. $25^x = 125$
3. $4^x = \frac{1}{8}$
4. $36^x = \sqrt{6}$
5. $3^x = \frac{1}{27}$
6. $5^x = \sqrt{125}$
7. $8^{2+x} = 2$
8. $4^{1-x} = 8$
9. $27^{2x-1} = 3$
10. $49^{x-2} = 7\sqrt{7}$
11. $4^{2x+5} = 16^{x+1}$
12. $3^{-(x+5)} = 9^{4x}$
13. $25^{2x} = 5^{x+6}$
14. $6^{x+1} = 36^{x-1}$
15. $10^{x-1} = 100^{4-x}$

Solutions

Rational Exponents

1. $\sqrt[3]{2}$	2. $(\sqrt{37})^3$	3. $\sqrt{x}$	4. $\sqrt[5]{a}$	5. $(\sqrt[3]{6})^4$	6. $(\sqrt[3]{6})^3$
7. $\frac{1}{\sqrt{7}}$	8. $\frac{1}{\sqrt[5]{9}}$	9. $\frac{1}{\sqrt[7]{x^3}}$	10. $\frac{1}{\sqrt[5]{b^6}}$	11. $\sqrt{3x}$	12. $3\sqrt{x}$
13. $7^{\frac{1}{2}}$	14. $34^{\frac{1}{2}}$	15. $(-11)^{\frac{1}{3}}$	16. $a^{\frac{2}{5}}$	17. $6^{\frac{4}{3}}$	18. $b^{\frac{4}{3}}$
19. $x^{-\frac{1}{2}}$	20. $a^{-\frac{1}{3}}$	21. $x^{-\frac{4}{5}}$	22. $2^{\frac{1}{3}}b$	23. $3^{\frac{1}{2}}x^{\frac{5}{2}}$	24. $5^{\frac{1}{4}}t^{\frac{3}{4}}$
25. 2	26. 5	27. $\frac{1}{2}$	28. -2	29. 5	30. $-\frac{1}{3}$
31. $\frac{1}{2}$	32. 0.2	33. 3	34. 0.1	35. $\frac{2}{3}$	36. $\frac{3}{2}$
37. 4	38. 8	39. 243	40. 27	41. $\frac{1}{8}$	42. 4
43. $-\frac{1}{32}$	44. $\frac{1}{9}$	45. 1	46. 1	47. $\frac{1000}{27}$	48. $\frac{4}{9}$

49. a) 3	b) $\frac{1}{3}$	50. a) 2	b) $\frac{1}{2}$	51. a) 4	b) $\frac{1}{4}$
52. a) 2	b) $\frac{1}{2}$	53. $\frac{1}{3}$	54. 9	55. 8	56. $\frac{1}{5}$
57. 9	58. $\frac{1}{7}$	59. 8	60. 8	61. $-\frac{1}{5}$	62. $\frac{1}{2}$
63. -4	64. $\frac{1}{5}$	65. $\frac{1}{2}$	66. 49		

Equations with Rational Exponents

1. 2 2. $\frac{3}{2}$ 3. -3 4. $-\frac{4}{3}$

5. a) 16 b) 5 6. a) $\frac{1}{36}$ b) $\frac{1}{108}$

7. a) $\frac{1}{25}$ b) $\frac{1}{200}$ 8. a) $\frac{1}{72}$ b) $\frac{27}{8}$

9. -56 10. 3 11. ± 11 12. ± 4

Exponential Equations

1. 1 2. $\frac{3}{2}$ 3. $\frac{1}{2}$ 4. $\frac{1}{4}$ 5. -3 6. $\frac{3}{2}$

7. $-\frac{5}{3}$ 8. $-\frac{1}{2}$ 9. $\frac{2}{3}$ 10. $\frac{11}{4}$ 11. No solution

12. $-\frac{5}{9}$ 13. 2 14. 3 15. 3