Advanced Mathematics 2200

Curriculum Guide 2017

Newfoundland Labrador

Education and Early Childhood Development

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Acknowledgements

The Department of Education and Early Childhood Development would like to thank the Western and Northern Canadian Protocol (WNCP) for Collaboration in Education, *The Common Curriculum Framework for K-9 Mathematics* - May 2006 and the Common Curriculum Framework for Grades 10-12 - January 2008, reproduced and/or adapted by permission. All rights reserved.

We would also like to thank the provincial Grade 11 Advanced Mathematics curriculum committee and the following people for their contribution:

Nicole Bishop, Program Development Specialist - Mathematics, Division of Programs and Services, Department of Education and Early Childhood Development

John-Kevin Flynn, Test Development Specialist - Mathematics/Science, Division of Evaluation and Research, Department of Education

Joanne Hogan, Program Development Specialist - Mathematics, Division of Program Development, Department of Education

Deanne Lynch, Program Development Specialist - Mathematics, Division of Program Development, Department of Education

Michelle Broders, Mathematics Teacher - Roncalli High School, Avondale

Mary Ellen Giles, Mathematics Teacher - Mealy Mountain Collegiate, Happy Valley-Goose Bay

Paul Gosse, Mathematics Teacher - CDLI

Dean Holloway, Mathematics Teacher - Heritage Collegiate, Lethbridge

Kelly House, Mathematics Teacher - Holy Heart of Mary High, St. John's

Brent Hughes, Mathematics Teacher - Exploits Valley High, Grand Falls-Windsor

Jennifer Lawlor, Mathematics Teacher - Baltimore School, Ferryland

Jonathan Payne, Mathematics Teacher - Corner Brook Regional High School, Corner Brook

Jim Penney, Mathematics Teacher - Stephenville High School, Stephenville

Deanne Rogers, Mathematics Teacher - Random Island Academy, Hickman's Harbour

Clarence White, Mathematics Teacher - Elwood Regional High, Deer Lake

Trevor Young, Mathematics Teacher - Exploits Valley High School, Grand Falls-Windsor

INTRODUCTION

Background

The curriculum guide communicates high expectations for students.

The Mathematics curriculum guides for Newfoundland and Labrador have been derived from *The Common Curriculum Framework for 10-12 Mathematics: Western and Northern Canadian Protocol*, January 2008. These guides incorporate the conceptual framework for Grades 10 to 12 Mathematics and the general outcomes, specific outcomes and achievement indicators established in the common curriculum framework. They also include suggestions for teaching and learning, suggested assessment strategies, and an identification of the associated resource match between the curriculum and authorized, as well as recommended, resource materials. This Mathematics 2200 course was originally implemented in 2012.

Beliefs About Students and Mathematics Learning

Mathematical understanding is fostered when students build on their own experiences and prior knowledge.

Students are curious, active learners with individual interests, abilities and needs. They come to classrooms with varying knowledge, life experiences and backgrounds. A key component in developing mathematical literacy is making connections to these backgrounds and experiences.

Students learn by attaching meaning to what they do, and they need to construct their own meaning of mathematics. This meaning is best developed when learners encounter mathematical experiences that proceed from the simple to the complex and from the concrete to the abstract. Through the use of manipulatives and a variety of pedagogical approaches, teachers can address the diverse learning styles, cultural backgrounds and developmental stages of students, and enhance within them the formation of sound, transferable mathematical understandings. Students at all levels benefit from working with a variety of materials, tools and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions provide essential links among concrete, pictorial and symbolic representations of mathematical concepts.

The learning environment should value and respect the diversity of students' experiences and ways of thinking, so that students feel comfortable taking intellectual risks, asking questions and posing conjectures. Students need to explore problem-solving situations in order to develop personal strategies and become mathematically literate. They must come to understand that it is acceptable to solve problems in a variety of ways and that a variety of solutions may be acceptable.

Program Design and Components

Affective Domain

To experience success, students must learn to set achievable goals and assess themselves as they work toward these goals.

A positive attitude is an important aspect of the affective domain and has a profound impact on learning. Environments that create a sense of belonging, encourage risk taking and provide opportunities for success help develop and maintain positive attitudes and self-confidence within students. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations and engage in reflective practices.

Teachers, students and parents need to recognize the relationship between the affective and cognitive domains, and attempt to nurture those aspects of the affective domain that contribute to positive attitudes. To experience success, students must learn to set achievable goals and assess themselves as they work toward these goals.

Striving toward success and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting, assessing and revising personal goals.

Goals For Students

Mathematics education must prepare students to use mathematics confidently to solve problems.

The main goals of mathematics education are to prepare students to:

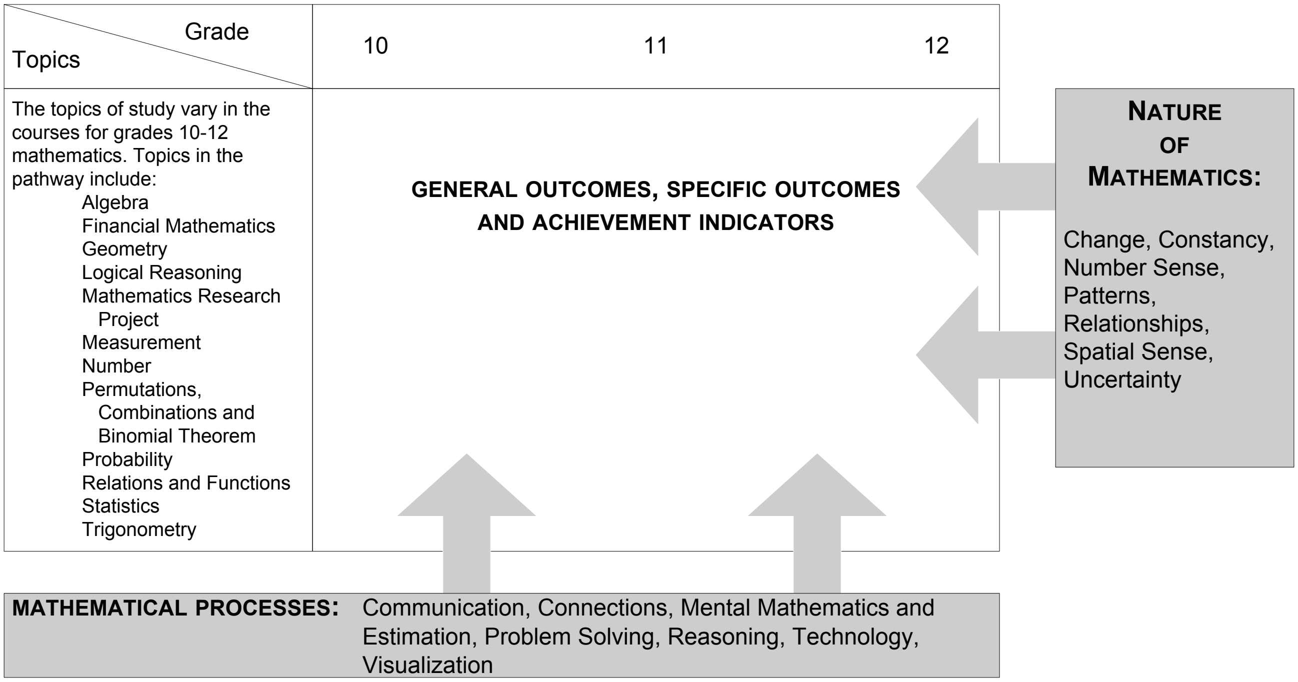
- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- commit themselves to lifelong learning
- become mathematically literate adults, using mathematics to contribute to society.

Students who have met these goals will:

- gain understanding and appreciation of the contributions of mathematics as a science, philosophy and art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity.

CONCEPTUAL FRAMEWORK FOR 10-12 MATHEMATICS

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.



Mathematical Processes

- *Communication [C]*
- *Connections [CN]*
- *Mental Mathematics and Estimation [ME]*
- *Problem Solving [PS]*
- *Reasoning [R]*
- *Technology [T]*
- *Visualization [V]*

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and embrace lifelong learning in mathematics.
Students are expected to:

- communicate in order to learn and express their understanding
- connect mathematical ideas to other concepts in mathematics, to everyday experiences and to other disciplines
- demonstrate fluency with mental mathematics and estimation
- develop and apply new mathematical knowledge through problem solving
- develop mathematical reasoning
- select and use technologies as tools for learning and for solving problems
- develop visualization skills to assist in processing information, making connections and solving problems.

This curriculum guide incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning.

Communication [C]

Students must be able to communicate mathematical ideas in a variety of ways and contexts.

Students need opportunities to read about, represent, view, write about, listen to and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics.

Communication is important in clarifying, reinforcing and modifying ideas, attitudes and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics. Students also need to communicate their learning using mathematical terminology.

Communication helps students make connections among concrete, pictorial, symbolic, oral, written and mental representations of mathematical ideas.

Connections [CN]

Through connections, students begin to view mathematics as useful and relevant.

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students begin to view mathematics as useful, relevant and integrated.

Learning mathematics within contexts and making connections relevant to learners can validate past experiences and increase student willingness to participate and be actively engaged.

The brain is constantly looking for and making connections. "Because the learner is constantly searching for connections on many levels, educators need to *orchestrate the experiences* from which learners extract understanding … Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching" (Caine and Caine, 1991, p.5).

Mental Mathematics and Estimation [ME]

Mental mathematics and estimation are fundamental components of number sense.

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external memory aids.

Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy and flexibility.

"Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental math" (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics "... become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers and are more able to use multiple approaches to problem solving" (Rubenstein, 2001, p. 442).

Mental mathematics "... provides the cornerstone for all estimation processes, offering a variety of alternative algorithms and nonstandard techniques for finding answers" (Hope, 1988, p. v).

Estimation is used for determining approximate values or quantities or for determining the reasonableness of calculated values. It often uses benchmarks or referents. Students need to know when to estimate, how to estimate and what strategy to use.

Estimation assists individuals in making mathematical judgments and in developing useful, efficient strategies for dealing with situations in daily life.

Problem Solving [PS]

Learning through problem solving should be the focus of mathematics at all grade levels.

Learning through problem solving should be the focus of mathematics at all grade levels. When students encounter new situations and respond to questions of the type, "How would you know?" or "How could you ...?", the problem-solving approach is being modelled. Students develop their own problem-solving strategies by listening to, discussing and trying different strategies.

A problem-solving activity requires students to determine a way to get from what is known to what is unknown. If students have already been given steps to solve the problem, it is not a problem, but practice. A true problem requires students to use prior learning in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement.

Problem solving is a powerful teaching tool that fosters multiple, creative and innovative solutions. Creating an environment where students openly seek and engage in a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk takers.

Reasoning [R]

Mathematical reasoning helps students think logically and make sense of mathematics.

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking. High-order questions challenge students to think and develop a sense of wonder about mathematics.

Mathematical experiences in and out of the classroom provide opportunities for students to develop their ability to reason. Students can explore and record results, analyze observations, make and test generalizations from patterns, and reach new conclusions by building upon what is already known or assumed to be true.

Reasoning skills allow students to use a logical process to analyze a problem, reach a conclusion and justify or defend that conclusion.

Technology [T]

Technology contributes to the learning of a wide range of mathematical outcomes and enables students to explore and create patterns, examine relationships, test conjectures and solve problems.

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Technology can be used to:

- explore and demonstrate mathematical relationships and patterns
- organize and display data
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- decrease the time spent on computations when other mathematical learning is the focus
- reinforce the learning of basic facts
- develop personal procedures for mathematical operations
- create geometric patterns
- simulate situations
- develop number sense.

Technology contributes to a learning environment in which the growing curiosity of students can lead to rich mathematical discoveries at all grade levels.

Visualization [V]

Visualization "involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world" (Armstrong, 1993, p. 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them.

Visualization is fostered through the use of concrete materials, technology and a variety of visual representations.

Visual images and visual reasoning are important components of number, spatial and measurement sense. Number visualization occurs when students create mental representations of numbers.

Being able to create, interpret and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure, when to estimate and which estimation strategies to use (Shaw and Cliatt, 1989).

Nature of Mathematics

- *Change*
- *Constancy*
- *Number Sense*
- *Patterns*
- *Relationships*
- *Spatial Sense*
- *Uncertainty*

Mathematics is one way of trying to understand, interpret and describe our world. There are a number of components that define the nature of mathematics and these are woven throughout this curriculum guide. The components are change, constancy, number sense, patterns, relationships, spatial sense and uncertainty.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics.

Change is an integral part of mathematics and the learning of mathematics.

Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, … can be described as:

- the number of a specific colour of beads in each row of a beaded design
- skip counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain

(Steen, 1990, p. 184).

Constancy

Constancy is described by the terms stability, conservation, equilibrium, steady state and symmetry.

Different aspects of constancy are described by the terms stability, conservation, equilibrium, steady state and symmetry (AAAS-Benchmarks, 1993, p.270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include the following:

- The ratio of the circumference of a teepee to its diameter is the same regardless of the length of the teepee poles.
- The sum of the interior angles of any triangle is 180°.
- The theoretical probability of flipping a coin and getting heads is 0.5.

Some problems in mathematics require students to focus on properties that remain constant. The recognition of constancy enables students to solve problems involving constant rates of change, lines with constant slope, direct variation situations or the angle sums of polygons.

Number Sense

An intuition about number is the most important foundation numeracy.

Number sense, which can be thought of as intuition about numbers, is the most important foundation of numeracy (British Columbia Ministry of Education, 2000, p.146).

A true sense of number goes well beyond the skills of simply counting, memorizing facts and the situational rote use of algorithms. Mastery of number facts is expected to be attained by students as they develop their number sense. This mastery allows for facility with more complex computations but should not be attained at the expense of an understanding of number.

Number sense develops when students connect numbers to their own real-life experiences and when students use benchmarks and referents. This results in students who are computationally fluent and flexible with numbers and who have intuition about numbers. The evolving number sense typically comes as a by product of learning rather than through direct instruction. It can be developed by providing rich mathematical tasks that allow students to make connections to their own experiences and their previous learning.

Patterns

Mathematics is about recognizing, describing and working with numerical and non-numerical patterns.

Mathematics is about recognizing, describing and working with numerical and non-numerical patterns. Patterns exist in all strands of mathematics.

Working with patterns enables students to make connections within and beyond mathematics. These skills contribute to students' interaction with, and understanding of, their environment.

Patterns may be represented in concrete, visual or symbolic form. Students should develop fluency in moving from one representation to another.

Students must learn to recognize, extend, create and use mathematical patterns. Patterns allow students to make predictions and justify their reasoning when solving routine and non-routine problems.

Learning to work with patterns in the early grades helps students develop algebraic thinking, which is foundational for working with more abstract mathematics.

Relationships

Mathematics is used to describe and explain relationships.

Mathematics is one way to describe interconnectedness in a holistic worldview. Mathematics is used to describe and explain relationships. As part of the study of mathematics, students look for relationships among numbers, sets, shapes, objects and concepts. The search for possible relationships involves collecting and analyzing data and describing relationships visually, symbolically, orally or in written form.

Spatial Sense

Spatial sense offers a way to interpret and reflect on the physical environment.

Spatial sense involves visualization, mental imagery and spatial reasoning. These skills are central to the understanding of mathematics.

Spatial sense is developed through a variety of experiences and interactions within the environment. The development of spatial sense enables students to solve problems involving 3-D objects and 2-D shapes and to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of shapes and objects. Spatial sense allows students to make predictions about the results of changing these dimensions; e.g., doubling the length of the side of a square increases the area by a factor of four. Ultimately, spatial sense enables students to communicate about shapes and objects and to create their own representations.

Uncertainty

In mathematics, interpretations of data and the predictions made from data may lack certainty.

Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty.

Uncertainty is an inherent part of making predictions.

The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation.

Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately.

Essential Graduation Learnings

Essential graduation learnings are statements describing the knowledge, skills and attitudes expected of all students who graduate from high school. Essential graduation learnings are cross-curricular in nature and comprise different areas of learning: *aesthetic expression, citizenship, communication, personal development, problem solving, technological competence* and *spiritual and moral development*.

Aesthetic Expression

Graduates will be able to respond with critical awareness to various forms of the arts and be able to express themselves through the arts.

Citizenship

Graduates will be able to assess social, cultural, economic and environmental interdependence in a local and global context.

Communication

Graduates will be able to use the listening, viewing, speaking, reading and writing modes of language(s) and mathematical and scientific concepts and symbols to think, learn and communicate effectively.

Personal Development

Graduates will be able to continue to learn and to pursue an active, healthy lifestyle.

Problem Solving

Graduates will be able to use the strategies and processes needed to solve a wide variety of problems, including those requiring language and mathematical and scientific concepts.

Technological Competence

Graduates will be able to use a variety of technologies, demonstrate an understanding of technological applications, and apply appropriate technologies for solving problems.

Spiritual and Moral Development

Graduates will be able to demonstrate an understanding and appreciation for the place of belief systems in shaping the development of moral values and ethical conduct.

See *Foundations for the Atlantic Canada Mathematics Curriculum*, pages 4-6.

The mathematics curriculum is designed to make a significant contribution towards students' meeting each of the essential graduation learnings (EGLs), with the communication, problem-solving and technological competence EGLs relating particularly well to the mathematical processes.

Outcomes and Achievement Indicators

The curriculum is stated in terms of general outcomes, specific outcomes and achievement indicators.

General Outcomes

General outcomes are overarching statements about what students are expected to learn in each course.

Specific Outcomes

Specific outcomes are statements that identify the specific skills, understanding and knowledge that students are required to attain by the end of a given course.
In the specific outcomes, the word *including* indicates that any ensuing items must be addressed to fully meet the learning outcome. The phrase *such as* indicates that the ensuing items are provided for illustrative purposes or clarification, and are not requirements that must be addressed to fully meet the learning outcome.

Achievement Indicators

Achievement indicators are samples of how students may demonstrate their achievement of the goals of a specific outcome. The range of samples provided is meant to reflect the scope of the specific outcome.

Specific curriculum outcomes represent the means by which students work toward accomplishing the general curriculum outcomes and ultimately, the essential graduation learnings.

Program Organization

<table>
<tr><th>Program Level</th><th>Course 1</th><th>Course 2</th><th>Course 3</th><th>Course 4</th></tr>
<tr><td>Advanced</td><td rowspan="2">Mathematics 1201</td><td>Mathematics 2200</td><td>Mathematics 3200</td><td>Mathematics 3208</td></tr>
<tr><td>Academic</td><td>Mathematics 2201</td><td>Mathematics 3201</td><td></td></tr>
<tr><td>Applied</td><td>Mathematics 1202</td><td>Mathematics 2202</td><td>Mathematics 3202</td><td></td></tr>
</table>

The applied program is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into the majority of trades and for direct entry into the workforce.

The academic and advanced programs are designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs. Students who complete the advanced program will be better prepared for programs that require the study of calculus.

The programs aim to prepare students to make connections between mathematics and its applications and to become numerate adults, using mathematics to contribute to society.

Summary

The conceptual framework for Grades 10-12 Mathematics (p. 3) describes the nature of mathematics, mathematical processes and the mathematical concepts to be addressed. The components are not meant to stand alone. Activities that take place in the mathematics classroom should result from a problem-solving approach, be based on mathematical processes and lead students to an understanding of the nature of mathematics through specific knowledge, skills and attitudes among and between topics.

ASSESSMENT AND EVALUATION

Purposes of Assessment

What learning is assessed and evaluated, how it is assessed and evaluated, and how results are communicated send clear messages to students and others about what is really valued.

Assessment techniques are used to gather information for evaluation. Information gathered through assessment helps teachers determine students' strengths and needs in their achievement of mathematics and guides future instructional approaches.

Teachers are encouraged to be flexible in assessing the learning success of all students and to seek diverse ways in which students might demonstrate what they know and are able to do.

Evaluation involves the weighing of the assessment information against a standard in order to make an evaluation or judgment about student achievement.

Assessment has three interrelated purposes:

- assessment *for* learning to guide and inform instruction;
- assessment *as* learning to involve students in self-assessment and setting goals for their own learning; and
- assessment *of* learning to make judgements about student performance in relation to curriculum outcomes.

Assessment *for* Learning

Assessment *for* learning involves frequent, interactive assessments designed to make student understanding visible. This enables teachers to identify learning needs and adjust teaching accordingly. It is an ongoing process of teaching and learning.

Assessment *for* learning:

- requires the collection of data from a range of assessments as investigative tools to find out as mush as possible about what students know
- provides descriptive, specific and instructive feedback to students and parents regarding the next stage of learning
- actively engages students in their own learning as they assess themselves and understand how to improve performance.

Assessment *as* Learning

Assessment *as* learning actively involves students' reflection on their learning and monitoring of their own progress. It focuses on the role of the student as the critical connector between assessment and learning, thereby developing and supporting metacognition in students.

Assessment *as* learning:

- supports students in critically analyzing their learning related to learning outcomes
- prompts students to consider how they can continue to improve their learning
- enables students to use information gathered to make adaptations to their learning processes and to develop new understandings.

Assessment *of* Learning

Assessment *of* learning involves strategies to confirm what students know, demonstrate whether or not they have met curriculum outcomes, or to certify proficiency and make decisions about students' future learning needs. Assessment *of* learning occurs at the end of a learning experience that contributes directly to reported results.

Traditionally, teachers relied on this type of assessment to make judgments about student performance by measuring learning after the fact and then reporting it to others. Used in conjunction with the other assessment processes previously outlined, however, assessment *of* learning is strengthened.

Assessment *of* learning:

- provides opportunities to report evidence to date of student achievement in relation to learning outcomes, to parents/guardians and other stakeholders
- confirms what students know and can do
- occurs at the end of a learning experience using a variety of tools.

Because the consequences of assessment *of* learning are often far-reaching, teachers have the responsibility of reporting student learning accurately and fairly, based on evidence obtained from a variety of contexts and applications.

Assessment Strategies

Assessment techniques should match the style of learning and instruction employed. Several options are suggested in this curriculum guide from which teachers may choose, depending on the curriculum outcomes, the class and school/district policies.

Observation (formal or informal)

This technique provides a way of gathering information fairly quickly while a lesson is in progress. When used formally, the student(s) would be aware of the observation and the criteria being assessed. Informally, it could be a frequent, but brief, check on a given criterion. Observation may offer information about the participation level of a student for a given task, use of a concrete model or application of a given process. The results may be recorded in the form of checklists, rating scales or brief written notes. It is important to plan in order that specific criteria are identified, suitable recording forms are ready, and all students are observed within a reasonable period of time.

Performance

This curriculum encourages learning through active participation. Many of the curriculum outcomes promote skills and their applications. In order for students to appreciate the importance of skill development, it is important that assessment provide feedback on the various skills. These may be the correct manner in which to use a manipulative, the ability to interpret and follow instructions, or to research, organize and present information. Assessing performance is most often achieved through observing the process.

Paper and Pencil

These techniques can be formative or summative. Whether as part of learning, or a final statement, students should know the expectations for the exercise and how it will be assessed. Written assignments and tests can be used to assess knowledge, understanding and application of concepts. They are less successful at assessing processes and attitudes. The purpose of the assessment should determine what form of paper and pencil exercise is used.

Journal

Journals provide an opportunity for students to express thoughts and ideas in a reflective way. By recording feelings, perceptions of success, and responses to new concepts, a student may be helped to identify his or her most effective learning style. Knowing how to learn in an effective way is powerful information. Journal entries also give indicators of developing attitudes to mathematical concepts, processes and skills, and how these may be applied in the context of society. Self-assessment, through a journal, permits a student to consider strengths and weaknesses, attitudes, interests and new ideas. Developing patterns may help in career decisions and choices of further study.

Interview

This curriculum promotes understanding and applying mathematics concepts. Interviewing a student allows the teacher to confirm that learning has taken place beyond simple factual recall. Discussion allows a student to display an ability to use information and clarify understanding. Interviews may be a brief discussion between teacher and student or they may be more extensive. Such conferences allow students to be proactive in displaying understanding. It is helpful for students to know which criteria will be used to assess formal interviews. This assessment technique provides an opportunity to students whose verbal presentation skills are stronger than their written skills.

Presentation

The curriculum includes outcomes that require students to analyze and interpret information, to be able to work in teams, and to communicate information. These activities are best displayed and assessed through presentations. These can be given orally, in written/pictorial form, by project summary, or by using electronic systems such as video or computer software. Whatever the level of complexity, or format used, it is important to consider the curriculum outcomes as a guide to assessing the presentation. The outcomes indicate the process, concepts and context for which a presentation is made.

Portfolio

Portfolios offer another option for assessing student progress in meeting curriculum outcomes over a more extended period of time. This form of assessment allows the student to be central to the process. There are decisions about the portfolio, and its contents, which can be made by the student. What is placed in the portfolio, the criteria for selection, how the portfolio is used, how and where it is stored, and how it is evaluated are some of the questions to consider when planning to collect and display student work in this way. The portfolio should provide a long-term record of growth in learning and skills. This record of growth is important for individual reflection and self-assessment, but it is also important to share with others. For all students, it is exciting to review a portfolio and see the record of development over time.

INSTRUCTIONAL FOCUS

Planning for Instruction

Consider the following when planning for instruction:

- Integration of the mathematical processes within each topic is expected.
- By decreasing emphasis on rote calculation, drill and practice, and the size of numbers used in paper and pencil calculations, more time is available for concept development.
- Problem solving, reasoning and connections are vital to increasing mathematical fluency and must be integrated throughout the program.
- There should be a balance among mental mathematics and estimation, paper and pencil exercises, and the use of technology, including calculators and computers. Concepts should be introduced using manipulatives and be developed concretely, pictorially and symbolically.
- Students bring a diversity of learning styles and cultural backgrounds to the classroom. They will be at varying developmental stages.

Teaching Sequence

The curriculum guide for Advanced Mathematics 2200 is organized by units. This is only a suggested teaching order for the course. There are a number of combinations of sequences that would be appropriate.

Each two page spread lists the topic, general outcome, and specific outcome.

Instruction Time Per Unit

The suggested number of hours of instruction per unit is listed in the guide at the beginning of each unit. The number of suggested hours includes time for completing assessment activities, reviewing and evaluating. The timelines at the beginning of each unit are provided to assist in planning. The use of these timelines is not mandatory. However, it is mandatory that all outcomes are taught during the school year, so a long term plan is advised. Teaching of the outcomes is ongoing, and may be revisited as necessary.

Resources

The authorized resource for Newfoundland and Labrador students and teachers is *Pre-Calculus 11* (McGraw-Hill Ryerson). Column four of the curriculum guide references *Pre-Calculus 11* for this reason.
Teachers may use any other resource, or combination of resources, to meet the required specific outcomes.

GENERAL AND SPECIFIC OUTCOMES

GENERAL AND SPECIFIC OUTCOMES WITH ACHIEVEMENT INDICATORS (pp. 19- 214)

This section presents general and specific outcomes with corresponding achievement indicators and is organized by unit. The list of indicators contained in this section is not intended to be exhaustive but rather to provide teachers with examples of evidence of understanding that may be used to determine whether or not students have achieved a given specific outcome. Teachers may use any number of these indicators or choose to use other indicators as evidence that the desired learning has been achieved. Achievement indicators should also help teachers form a clear picture of the intent and scope of each specific outcome.

Advanced Mathematics 2200 is organized into nine units: *Trigonometry, Quadratic Functions, Quadratic Equations, Radical Expressions and Equations, Rational Expressions and Equations, Absolute Value and Reciprocal Functions, Systems of Equations, Linear and Quadratic Inequalities,* and *Sequences and Series.*

Trigonometry

Suggested Time: 14 Hours

Unit Overview

Focus and Context

In this unit, students will be exposed to angles in standard position where $0\ ° \leq \theta \leq 360°$. They will also be introduced to reference angles and quadrantal angles. Students will determine the exact values of the primary trigonometric ratios for angles between 0° and 360°.

Students will use the sine law and the cosine law to determine unknown side lengths and angle measures in oblique triangles. They will be asked to explain their thinking about which law to use.

Outcomes Framework

GCO

Develop trigonometric reasoning.

SCO T1

Demonstrate an understanding of angles in standard position [0° to 360°].

SCO T2

Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position.

SCO T3

Solve problems, using the cosine law and sine law, including the ambiguous case.

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Measurement	Trigonometry	Trigonometry
M4 Develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles. [C, CN, PS, R, T, V]	T1 Demonstrate an understanding of angles in standard position [0° to 360°]. [R, V] T2 Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position. [C, ME, PS, R, T, V] T3 Solve problems, using the cosine law and sine law, including the ambiguous case. [C, CN, PS, R, T]	T1 Demonstrate an understanding of angles in standard position, expressed in degrees and radians. [CN, ME, R, V] T2 Develop and apply the equation of the unit circle. [CN, R, V] T3 Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees. [ME, PS, R, T, V] T4 Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems. [CN, PS, T, V] T5 Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians. [CN, PS, R, T, V] T6 Prove trigonometric identities, using: • reciprocal identities • quotient identities • Pythagorean identities • sum or difference identities (restricted to sine, cosine and tangent) • double-angle identities (restricted to sine, cosine and tangent). [R, T, V]

Mathematical Processes

[C]	Communication	[PS]	Problem Solving
[CN]	Connections	[R]	Reasoning
[ME]	Mental Mathematics and Estimation	[T]	Technology
		[V]	Visualization

Trigonometry

Outcomes

Students will be expected to

T1 Demonstrate an understanding of angles in standard position [0° to 360°].

[R, V]

Achievement Indicators:

T1.1 *Sketch an angle in standard position, given the measure of the angle.*

T1.2 *Determine the quadrant in which a given angle in standard position terminates.*

T1.3 *Determine the reference angle for an angle in standard position.*

T1.4 *Explain, using examples, how to determine the angles from* 0° to 360° *that have the same reference angle as a given angle.*

T1.5 *Illustrate, using examples, that any angle from 90° to 360° is the reflection in the x-axis and/or the y-axis of its reference angle.*

Elaborations—Strategies for Learning and Teaching

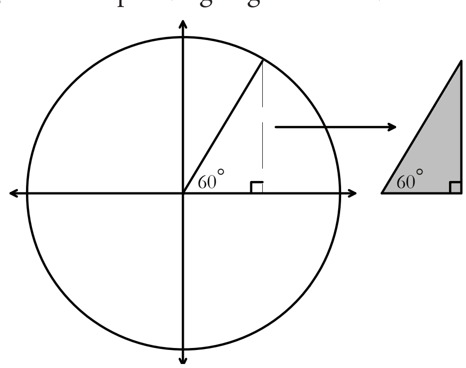
In Mathematics 1201, students used the Pythagorean theorem and the primary trigonometric ratios to find missing side lengths and angle measures in right triangles. They also investigated and worked with angles of elevation and depression (M4). In this unit, students will evaluate the primary trigonometric ratios for angles from 0° to 360° using the coordinate plane. They will also use the primary trigonometric ratios to solve problems involving oblique triangles, using the Law of Sines (including the ambiguous case) and the Law of Cosines.

In this unit, students will solve problems involving angles with measures between 0° and 360°. Negative rotational angles and co-terminal angles are not part of this outcome. They will be addressed in Mathematics 3200.

Introduce students to terminology such as the initial arm, terminal arm, vertex and standard position. They will sketch angles in standard position on the coordinate plane and identify the quadrant where the terminal arm lies.

An understanding of reference angles will be critical when students compare the trigonometric ratio of an angle in standard position to the trigonometric ratio of its reference angle. It will also be useful when determining the exact trigonometric ratios for angles in standard position that are multiples of 30°, 45°and 60°, and solving equations of the form sin $\theta = a$ or cos $\theta = a$.

To help students visualize the relationship between an angle sketched in standard position and its corresponding reference angle, provide students, individually or in small groups, with a copy of a circle sketched on grid paper. Give them a coloured cut out of the right triangle showing the corresponding angle sketched in standard position.



Guide students using the following directions and questions:

- Place the right triangle over the angle on the circle. What do you notice about the two angles? They should verify the 60° angle is in standard position.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Observation

- Create a model of the coordinate grid with an initial arm and terminal arm that can be moved physically. Ask students to demonstrate the placement of the terminal arm when given the measure of an angle between 0° and 360°. They should explore acute, obtuse, right, straight and reflex angles .

(T1.1)

Paper and Pencil

- Provide students with angles sketched in standard position from Quadrants II, III and IV, such as 150°, 210°, and 330°. Ask them to determine the related reference angle and explain why the reference angle for all three is the same.

(T1.1, T1.2, T1.3, T1.4)

- Ask students to determine the reference angle for each of the following angles:

  (i) 100°
  (ii) 250°
  (iii) 315°

(T1.3)

Resources/Notes

Authorized Resource

Pre-Calculus 11

2.1 Angles in Standard Position

Student Book (SB): pp. 74-87
Teacher Resource (TR): pp. 57-64

Blackline Master (BLM): 2-3, 2-4

Trigonometry

Outcomes

Students will be expected to

T1 Continued ...

Achievement Indicators:

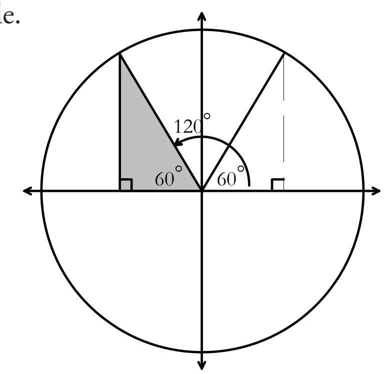
T1.3, T1.4, T1.5 *Continued*

T1.6 *Draw an angle in standard position given any point P(x, y) on the terminal arm of the angle.*

T1.7 *Illustrate, using examples, that the points P(x, y), P(-x, y), P(-x, -y)and P(x, -y) are points on the terminal arms of angles in standard position that have the same reference angle.*

Elaborations—Strategies for Learning and Teaching

- Reflect the triangle across the *y*-axis into Quadrant II.
- Determine the measure of the Quadrant II angle that is formed with the positive *x*-axis and the hypotenuse (terminal arm) of this triangle.



- Explain how the two angles are related.

Students should recognize that 120° = 180° - 60° or $\theta_R = 180° - \theta°$.

- Continue this activity in Quadrant III by asking students to rotate the 60° original angle 180° about the origin. Then ask students to reflect the original angle across the *x*-axis into Quadrant IV. Encourage students to explain their observations.

Students should observe that in Quadrant III, the two angles have the relationship 240° = 180° + 60° or $\theta_R = 180° + \theta°$. In Quadrant IV, the angles are 300° = 360°- 60° or $\theta_R = 360° - \theta°$.

Now that students have explored all quadrants, this would be a good opportunity to question them about the general properties of reference angles. Students should notice that the reference angle is always positive and measures between 0° and 90°.

Once students have sketched an angle in standard position and determined the quadrant in which it terminates, it is a natural extension to draw an angle given a point on its terminal arm. Students will determine which quadrant the terminal arm of the angle is located based on the point P(*x*, *y*) given.

Students should be given an opportunity to graph a point and reflect it on the *y*-axis, on the *x*-axis and across the origin. The various reflections will illustrate how a point on the terminal arm of (1, 2), for example, reflects to become (-1, 2) in quadrant II, (-1, -2) in quadrant III and (1, -2) in quadrant IV.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Interview

- Provide students with examples representing the endpoint of the terminal arm for angles sketched in standard position from Quadrants II, III and IV (e.g., (-3, 4), (-3, -4) and (3, -4)). Ask them to explain why the related reference angle for all three is the same.

(T1.1, T1.2, T1.3, T1.4)

Observation

- Ask students to explore the endpoint of the terminal arm of θ if it lies on the axis (i.e., (1, 0)(0, 1), (-1, 0) and (0, -1)).

(T1.6, T1.7)

Performance

- Ask students to create a "Four Door Book" foldable. On the front of the foldable, they should write the quadrants. On the inside of the foldable, students write the general "rule" for generating the reference angle in each quadrant and provide an example.

Front

Quadrant II	Quadrant I
Quadrant III	Quadrant IV

Inside

$\theta_R = 180° - \theta$	θ_R
$\theta_R = 180° + \theta$	$\theta_R = 360° - \theta$

(T1.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11

2.1 Angles in Standard Position

SB: pp. 74-87
TR: pp. 57-64
BLM: 2-3, 2-4

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/trig.html

- Four Door Book foldable

Trigonometry

Outcomes

Students will be expected to

T2 Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position.

[C, ME, PS, R, T, V]

Achievement Indicators:

T2.1 *Determine, using the Pythagorean theorem, the distance from the origin to a point P(x, y) on the terminal arm of an angle.*

T2.2 *Determine the value of sin θ, cos θ, or tan θ given any point P(x, y) on the terminal arm of angle θ .*

T2.3 *Determine the sign of a given trigonometric ratio for a given angle, without the use of technology, and explain.*

T2.4 *Sketch a diagram to represent a problem.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students applied the primary trigonometric ratios to angles between 0° and 90° (M4). They will now explore angles between 0° and 360° using coordinates and reference angles.

If θ is an angle in standard position, and point P(x, y) is a point on the terminal arm of angle θ, students will use the Pythagorean theorem to determine the length of the hypotenuse r. The three primary trigonometric ratios will be defined in terms of x, y and r.

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{y}{r} \qquad \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{x}{r} \qquad \tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{y}{x}$$

Students will work with exact trigonometric ratios which may require finding exact values for the hypotenuse. Special angles formed by the intersection of the x-axis and y-axis will also be investigated (i.e., quadrantal angles). Students will be expected to solve simple trigonometric equations of the form sin θ = a or cos θ = a, where $-1 \leq a \leq 1$, and tan θ = a, where a is a real number.

An activity such as the following provides an opportunity for students to explore the trigonometric ratios given any point on the terminal arm of angle θ. Begin with the point (-4, -5). Continue the activity to ensure that endpoints from all quadrants are used. Use the following directions and questions to guide students:

- Plot the given point P(-4, -5). What quadrant does this point lie in?
- Construct the corresponding angle in standard position.
- Drop a perpendicular to the x-axis creating a right triangle. Which value represents the adjacent side? Which value represents the opposite side?
- How can you determine the exact length of the hypotenuse?
- State the cosine, sine and tangent ratios associated with the angle.
- What determines the sign of the ratio? Explain your reasoning.

If θ is an angle in standard position, students should be able to summarize in which quadrant(s) the terminal arm of θ will lie if:

(i) tan θ > 0 or tan θ < 0

(ii) sin θ > 0 or sin θ < 0

(iii) cos θ > 0 or cos θ < 0

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to complete the following table and suggest strategies for determining the sign (±) of the various ratios in each of the quadrants. They should share and compare their findings with the rest of the class.

	cos θ	sin θ	tan θ
I			
II			
III			
IV			

(T2.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11

2.2 Trigonometric Ratios of Any Angle

SB: pp. 88-99
TR: pp. 65-72
BLM: 2-3, 2-5

Trigonometry

Outcomes

Students will be expected to

T2 Continued...

Achievement Indicators:

T2.5 *Determine, without the use of technology, the value of sin θ, cos θ, or tan θ given any point P(x, y) on the terminal arm of angle θ, where θ = 0 °, 90 °, 180 °, 270 °or 360 ° .*

T2.4 *Continued*

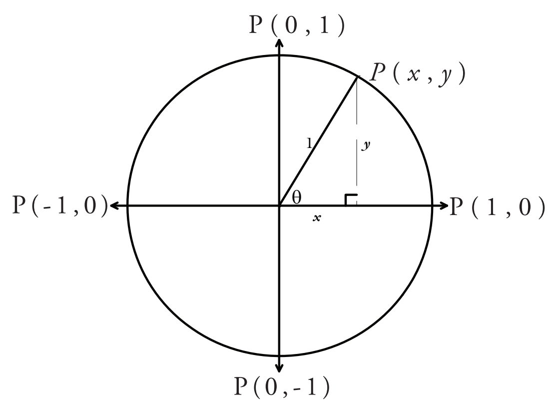
Elaborations—Strategies for Learning and Teaching

Trigonometric ratios for angles whose measurements are 0°, 90°, 180°, 270° or 360° will now be explored. Ask students to choose an angle and then sketch it on the coordinate plane. Ask them the following questions using the 90° angle as a guide:

- Where is the terminal arm?
- Does this angle have a corresponding reference angle?
- Can a reference triangle be drawn? Explain.

Students should recognize that every angle drawn in standard position has a corresponding reference angle, except for quadrantal angles.

While the unit circle is not formally introduced until Mathematics 3200, this would be an effective strategy to use when discussing quadrantal angles. The unit circle, where the radius equals one and the centre is (0, 0), can help students understand facts about the sine and cosine function.



Encourage students to use the diagram to help them write their definitions for sine, cosine and tangent for each quadrantal angle. Since $\cos\theta = \frac{x}{r} = \frac{x}{1} = x$ and $\sin\theta = \frac{y}{r} = \frac{y}{1} = y$ then P(*x*, *y*) = (cos θ, sin θ).

Challenge students to further explore this activity using the points (3, 0), (0, 3), (-3, 0) and (0, -3). Ask them how the length of the terminal arm affects the values of the cosine and sine ratios.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Observation

- Ask students to plot the points (1,0), (0,1), (-1,0) and (0,-1), and evaluate, using technology, the values of the sine and cosine ratio for 0°, 90°, 180°, 270° or 360°. They should explain any patterns or relationships that appear relating the *x* and *y* coordinate of each endpoint to the value of each trigonometric ratio.

(T2.4, T2.5)

Resources/Notes

Authorized Resource

Pre-Calculus 11

2.2 Trigonometric Ratios of Any Angle

SB: pp. 88-99
TR: pp. 65-72
BLM: 2-3, 2-5

Trigonometry

Outcomes

Students will be expected to

T2 Continued ...

Achievement Indicators:

T2.6 *Solve, for all values of θ, an equation of the form sin θ = a or cos θ = a, where $-1 \leq a \leq 1$, and an equation of the form tan θ = a, where a is a real number.*

T2.7 *Determine the exact value of the sine, cosine or tangent of a given angle with a reference angle of 30°, 45° or 60°.*

T2.4 *Continued*

Elaborations—Strategies for Learning and Teaching

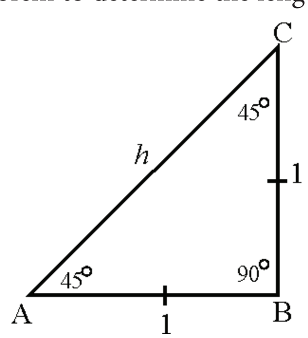
Previously, students were given an angle from 0° to 360° and asked to find the trigonometric value. They will now work backwards to find the missing angle when given an equation of the form sin θ = *a* or cos θ = *a,* where $-1 \leq a \leq 1$, and the equation of the form tan θ = *a,* where *a* is a real number. Use the following questions to guide them through the solution of an equation such as cos θ = 0.25:

- Determine the quadrants which contain solutions. Are there any restrictions?
- Determine the reference angle for the given value of *a* = 0.25.
- Determine the measure of the related angles in standard position where 0° ≤ θ ≤ 360°.

Students will be exposed to problems where they will have to rearrange the equation before they solve it. Ask them how they might solve an equation such as -2cos θ - 1 = 0 by comparing it to the linear equation -2*x* - 1 = 0. They should explain how the process is similar and how it is different.

Students have calculated the trigonometric values for angles between 0° and 360°. They have also discovered relationships that exist between the angle and its reference angle. For example, sin θ = - sin(θ_R) where θ is an angle in the third quadrant. Reference angles of 30°, 45° or 60° occur frequently in problems. It is equally important for students to understand where the sine, cosine or tangent value of these angles come from.

Geometric properties of right triangles containing 30°, 45° and 60° can be used to obtain the trigonometric values. To demonstrate the exact trigonometric value for θ = 45°, students can construct an isosceles triangle with the smallest side measuring 1 unit. They can then use the Pythagorean theorem to determine the length of the hypotenuse, $\sqrt{2}$.



General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to solve the equations for $0° \leq \theta \leq 360°$.

  (i) $\sin\theta = \frac{1}{3}$

  (ii) $\cos\theta = -\frac{3}{7}$

  (iii) $\tan\theta = \sqrt{3}$

  (iv) $\cos\theta = 0.8660$

  (v) $\sin\theta = 0.7071$

(T2.6)

Resources/Notes

Authorized Resource

Pre-Calculus 11

2.2 Trigonometric Ratios of Any Angle

SB: pp. 88-99
TR: pp. 65-72
BLM: 2-3, 2-5

Trigonometry

Outcomes

Students will be expected to

T2 Continued ...

Achievement Indicators:

T2.4, T2.7 *Continued*

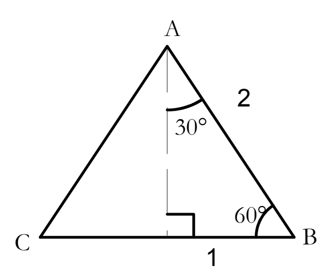
Elaborations—Strategies for Learning and Teaching

Consider the following guiding questions:

- What is the exact value of sin 45°?
- What do you notice if you evaluate sin 45° with a calculator?
- What do you notice about the values of sin 45° and cos 45°. Can you explain why they have the same value?
- What is the value of tan 45°? Can you determine the value of tan 90° using the right triangle? Explain your reasoning.
- What is the same and what is different about sin 45° and sin 225°?
- What is the same and what is different about cos 45° and cos 135°?
- What is the same and what is different about tan 45° and tan 135°?

When determining the exact value of the trigonometric ratios of a given angle, students will sometimes result in a value where the denominator contains a radical. Consider the example, cos 45°= $\frac{1}{\sqrt{2}}$. Students will not be expected to rationalize the denominator since the Radicals Unit is not done until later in this course.

A similar strategy to demonstrate the exact trigonometric values for θ = 30° or θ = 60° is for students to construct an equilateral triangle with a side length of 2. Drawing an altitude from vertex A, students can then use the Pythagorean theorem to determine the length of the altitude, $\sqrt{3}$.



General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Performance

- Divide the class into two groups. Individual students in one group will be given a card containing trigonometric ratios of special angles (i.e., sin 60°, cos 120°, etc). In the other group, students should be given associated exact values (i.e., $\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}$, etc). Ask students to find a partner to form a matching pair.

(T2.4, T2.7)

Journal

- When evaluating sin 45°, ask students to explain why $\frac{1}{\sqrt{2}}$ is called an exact value while 0.707 is approximate.

(T2.4, T2.7)

Interview

- Ask students to respond to the following questions when working with a 30°-60°-90° triangle.
  (i) Calculate the exact value of sin 30° and cos 30°.
  (ii) Calculate the exact value of sin 60° and cos 60°.
  (iii) Determine sin 30°, cos 30°, sin 60° and cos 60° using a calculator. What do you notice?
  (iv) Which is greater, sin 60° or sin 30°? Why?
  (v) What is the same and what is different about sin 30° and sin 150°?
  (vi) What is the same and what is different about cos 30° and cos 150°?
  (vii) What do you notice when you compare the exact value of tan 30° and tan 60° using the right triangle?
  (viii) What is the same and what is different about tan 30° and tan 330°?

(T2.4, T2.7)

Resources/Notes

Authorized Resource

Pre-Calculus 11

2.2 Trigonometric Ratios of Any Angle

SB: pp. 88-99
TR: pp. 65-72
BLM: 2-3, 2-5

Note: The ratios are often left in irrational form in the resource. (i.e., $\sin 45° = \frac{1}{\sqrt{2}}$)

Trigonometry

Outcomes

Students will be expected to

T2 Continued ...

Achievement Indicators:

T2.8 *Describe patterns in and among the values of the sine, cosine and tangent ratios for angles from to 0° to 360°.*

T2.9 *Solve a contextual problem, using trigonometric ratios.*

T2.4 *Continued*

Elaborations—Strategies for Learning and Teaching

Encourage students to use mental math to determine the trigonometric function values whenever the terminal side makes a 30°, 45° and 60° angle with the *x*-axis.

Students should be given an opportunity to explore the patterns in the sine, cosine and tangent ratios. They could use graphing technology and a table to record the points. Remind students to choose an appropriate increment for θ when graphing $y = \sin\theta$, $y = \cos\theta$ and $y = \tan\theta$ from 0° to 360°.

Students will be expected to solve problems using the trigonometric ratios. Encourage students to draw a sketch of a diagram to help them gain a visual understanding of the problem. Consider an example such as the following:

- The arm of a crane used for lifting very heavy objects can move so that it has a minimum angle of inclination of 30° and a maximum of 60°. Use exact values to find an expression for the change in the vertical displacement of the end of the arm, in terms of the length of the arm, *a*.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Observation

- Observe students graphing $y = \sin \theta$, $y = \cos \theta$ and $y = \tan \theta$ from 0° to 360°. Ask them to answer the following:
  - (i) What is the maximum value of $\sin \theta$, $\cos \theta$ and $\tan \theta$? the minimum value?
  - (ii) Where does the maximum value occur? the minimum value?
  - (iii) What are the *x*-intercepts and *y*-intercepts?
  - (iv) For what values of θ is the function positive? For what values is it negative?
  - (v) What comparisons can you make between the sine and cosine function?

(T2.8)

- Invite students to play WODB (Which One Doesn't Belong?). Show students the following expressions. Each expression could be the one which doesn't belong, but for a different reason. Observe students' reasoning for misconceptions.

- sin 30°	- sin 330°
cos 60°	sin 45°

As students become more familiar with WODB activities, challenge students to complete the following WODB. Observe students' reasoning and engage the class in a discussion about the variety of answers possible.

sin 90°	?
sin -630°	sin 405°

(T2.8)

Resources/Notes

Authorized Resource

Pre-Calculus 11

2.2 Trigonometric Ratios of Any Angle

SB: pp. 88-99
TR: pp. 65-72

BLM: 2-3, 2-5

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/trig.html

- WODB activities

Trigonometry

Outcomes

Students will be expected to

T3 Solve problems, using the cosine law and the sine law, including the ambiguous case.

[C, CN, PS, R, T]

Achievement Indicators:

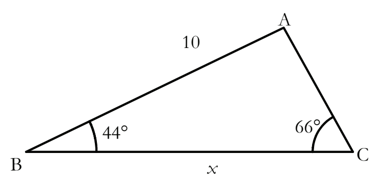
T3.1 *Sketch a diagram to represent a problem that involves a triangle without a right angle.*

T3.2 *Solve, using primary trigonometric ratios, a triangle that is not a right triangle.*

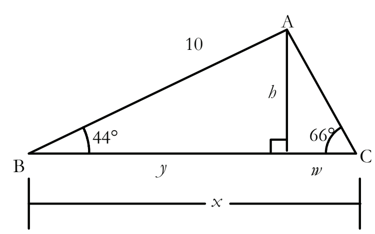
Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students used the three primary trigonometric ratios to determine the side lengths and angle measures in right triangles (M4). Solving problems involving trigonometry is rarely limited to right triangles. In this unit, students will derive the Law of Sines and the Law of Cosines and utilize them in a number of problem-solving situations.

Students have been exposed to right-triangle trigonometry to solve problems involving right triangles. They will now solve oblique triangles. Encourage students to draw a diagram to help them gain a visual understanding of the problem. Consider the following example:



Ask students if this triangle can be divided into two right triangles and what strategies can be applied to find the indicated side length. They should recognize that this requires a multi-step solution. A strategy must be developed before a solution is attempted. Drawing an altitude from vertex A, students can use the primary trigonometric ratios and the Pythagorean theorem to solve for the unknown value.



$\cos 44° = \frac{y}{10}\ (y = 7.2)$

$10^2 = 7.2^2 + h^2 (h = 6.9)$

$\tan 66° = \frac{6.9}{w}\ (w = 3.1)$

$x = y + w$

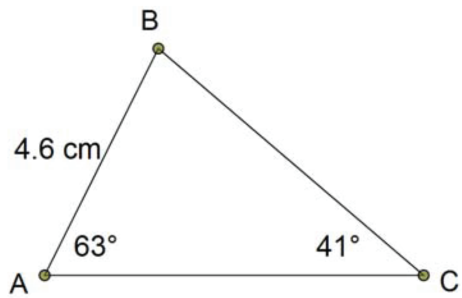
Students will be introduced to methods that are more efficient when solving an oblique triangle, namely the sine law and the cosine law.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to determine the length of AC in the following diagram:



(T3.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11

2.3 The Sine Law

SB: pp. 100-113
TR: pp. 73-80
BLM: 2-3, 2-7

Trigonometry

Outcomes

Students will be expected to

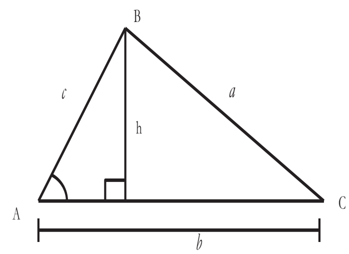
T3 Continued ...

Achievement Indicator:

T3.3 *Explain the steps in a given proof of the sine law and cosine law.*

Elaborations—Strategies for Learning and Teaching

Students should be guided through the derivation of the sine law and then use the law to solve triangles. The Law of Sines can be derived using right triangle trigonometry. Consider the following diagram:



△ABC is not a right triangle. Therefore, students will draw an altitude from vertex B. Ask students to write an expression for the height (h) using the sine ratio. Since $\sin A = \frac{h}{c}$, then $h = c$ (sin A). Similarly, since $\sin C = \frac{h}{a}$, then $h = a$ (sin C). Since both equations equal h, students can conclude that c (sin A) = a (sin C). Dividing both sides of the equation by sin A sin C gives:

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

By repeating these steps with other altitudes in △ABC, it can be concluded that:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Therefore,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

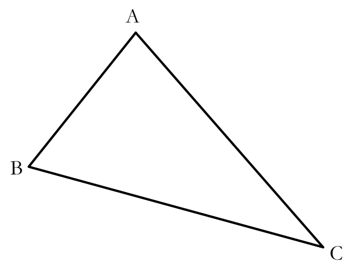
In other words, the sine law is a proportion that compares the ratio of each side of a triangle to its included angle.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Provide students with a triangle and have them measure the side lengths and angles using a ruler and protractor.



	Measure		Measure		Calculate
$\angle A$		side a		$\frac{\sin A}{a}$	
$\angle B$		side b		$\frac{\sin B}{b}$	
$\angle C$		side c		$\frac{\sin C}{c}$	

Ask students to answer the following questions:

(i) What can you conclude regarding the ratios calculated above?

(ii) Would your conclusion be valid if you were to use the reciprocal of the ratios?

(T3.3)

Resources/Notes

Authorized Resource

Pre-Calculus 11

2.3 The Sine Law

SB: pp. 100-113
TR: pp. 73-80
BLM: 2-3, 2-7

Trigonometry

Outcomes

Students will be expected to

T3 Continued ...

Achievement Indicator:

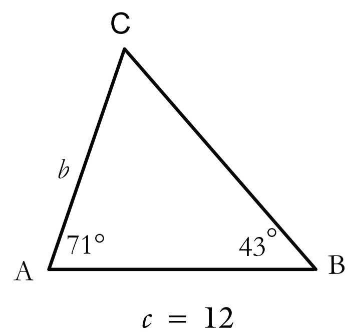
T3.4 *Sketch a diagram and solve a problem, using the sine law.*

Elaborations—Strategies for Learning and Teaching

Students will apply the sine law to determine unknown lengths and angle measures in triangles. Encourage them to draw diagrams with both the given and unknown information marked.

Ask students what information is needed to solve problems using the sine law. The Law of Sines involves a ratio of the sine of an angle to the length of its opposite side. Students should recognize that it will not work if no angle of the triangle is known or if one angle and its opposite side is not given.

When working with the sine law, students sometimes incorrectly identify side and opposite angle pairs. To avoid this error, encourage them to use arrows on the diagram when identifying the angle and its opposite side. They could also encounter problems when they cross multiply the ratios. When solving $\frac{\text{s inA}}{12} = \frac{\sin 30°}{4}$, for example, students may incorrectly write 4sinA = sin 360°. Teachers should emphasize the use of brackets and write 4(sinA)= 12(sin 30°). Another common student error occurs when students try to solve a triangle given two angles and an included side and mistakenly think there is not enough information to use the sine law. Consider an example such as the following:



Students can use the property that the sum of the angles in a triangle is 180°. Therefore, the measure of $\angle$C is 66°. They can then proceed to use the sine law to find the length of side AC. Encourage students to check the reasonableness of their answer. For example, since $\angle$ C is a little smaller than $\angle$ A, we expect the length of side AB to be a little shorter than the length of side CB. Students should also consider asking questions such as: Is the shortest side opposite the smallest angle? Is the longest side opposite the largest angle?

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

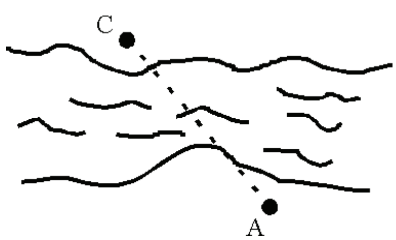
Observation

- Provide students with several practice problems using the sine law. As teachers observe students working through the problems, ask them the following questions:
  (i) What is the unknown? Is it an angle or a side?
  (ii) How can you isolate the unknown?
  (iii) How can you complete the calculations?
  (iv) How do you know whether to determine the sine or the inverse of the sine?
  (v) Does your conclusion answer the question asked?

(T3.4)

Interview

- A surveyor is located on one side of a river that is impossible to cross and only has a 100 m measuring tape and a sextant (used to measure angles) in his possession. Ask students to explain how the surveyor could use only these two tools and the Law of Sines to find the distance from point A to point C.



(T3.4)

Paper and Pencil

- Ask students to solve the following:
  The leaning tower of Pisa was originally perpendicular to the ground and stood 60 m. It's actual height now is 56.7 m. Because of shifting ground, the tower has now sunk and leans at a angle. When the top of the tower is seen from a point 46 m from its base, the angle of elevation is 53°. Determine the approximate angle at which the tower varies from the perpendicular.

(T3.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11

2.3 The Sine Law

SB: pp. 100-113
TR: pp. 73-80
BLM: 2-3, 2-7

Trigonometry

Outcomes

Students will be expected to

T3 Continued ...

Achievement Indicator:

T3.5 *Describe and explain situations in which a problem may have no solution, one solution or two solutions.*

Elaborations—Strategies for Learning and Teaching

When using the sine law, students will be exposed to the following situations:

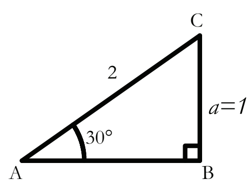
(i) Provided with two angles and an included side (ASA)

(ii) Provided with two angles and a non-included side (AAS)

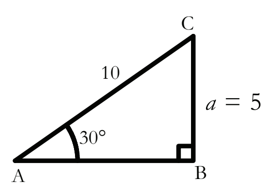
(iii) Provided with two sides and a non-included angle (SSA) – the ambiguous case

The ambiguous case may cause difficulty for some students. Provided with two sides and an angle opposite one of those sides, students generally expect that one triangle will result. A numerical example, outlined below, could help students further investigate the Law of Sines and observe special situations where one, two or no triangle is in fact possible.

Students are familiar with the 30°-60°-90° triangle.

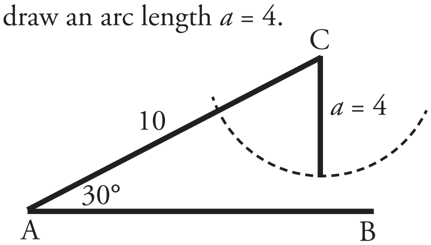


Using C as the center of the circle, ask students to construct ∠A = 30° and length AC = 10. Students should first notice that one triangle is possible where ∠B =90° and a = 5. They can verify this using the Law of Sines.



$$\frac{\sin 30°}{5} = \frac{\sin B}{10}$$
$$\sin B = \frac{10(\sin 30°)}{5}$$
$$\sin B = 1$$
$$\angle B = 90°$$

Students should then consider the case where a < 5. Using geometry, draw an arc length a = 4.



$$\frac{\sin 30°}{4} = \frac{\sin B}{10}$$
$$\sin B = \frac{10(\sin 30°)}{4}$$
$$\sin B = 1.25$$

Students should notice that no triangle is physically possible. If they do not attempt this approach, they may use the Law of Sines to try and solve this problem. This is an opportunity for students to review the maximum and minimum values of $y = \sin \theta$. Students should recognize that no triangle exists since the value of sin B > 1.

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to solve each of the following triangles using the Law of Sines (if possible) indicating which results in one triangle, two triangles or no triangle.

  (i) $\angle A = 45°$, $\angle B = 60°$, and $a = 14$

  (ii) $\angle B = 40°$, $a = 12$, $b = 6$

  (iii) $\angle A = 25°$, $a = 2$, $b = 3$

(T3.5)

Resources/Notes

Authorized Resource

Pre-Calculus 11

2.3 The Sine Law

SB: pp. 100-113
TR: pp. 73-80
BLM: 2-3, 2-7

Trigonometry

Outcomes

Students will be expected to

T3 Continued ...

Achievement Indicators:

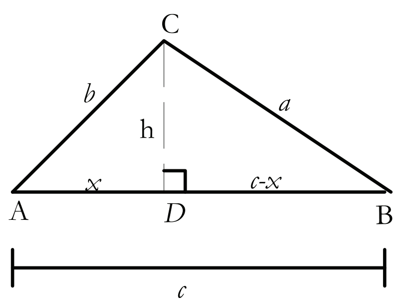
T3.5 *Continued*

Elaborations—Strategies for Learning and Teaching

Ask students to repeat this activity where $a > 5$. Using an arc length of $a = 8$, for example, the Law of Sines will yield two possible values for angle B (i.e., $\angle B = 53.1°$ or $\angle B = 126.9°$) and therefore, two possible triangles exist. Students are expected to use the Law of Sines to solve triangles where the ambiguous case exists. They are not expected to determine the number of triangles possible without solving.

T3.3 *Continued*

Students should be guided through the derivation of the cosine law and use it to solve triangles. Ask students to consider the triangle *ABC* with side lengths a, b and c . Draw an altitude, h, from vertex C and let D be the intersection of AB and the altitude as shown in the figure below. If x is the length of AD, then students should recognize that BD = $c - x$. Guide students through the following process:



- Use the Pythagorean theorem in ΔBCD : $a^2 = h^2 + (c - x)^2$
- Expand the binomial: $a^2 = h^2 + c^2 - 2cx + x^2$
- Apply the Pythagorean theorem in ΔADC ($x^2 + h^2 = b^2$). What do the expanded binomial yield if $b^2 - x^2$ is substituted for h^2. They should recognize the equation will result in $a^2 = b^2 + c^2 - 2cx$.
- What primary trigonometric ratio can be used to determine the altitude? They should recognize that in ΔACD, $\cos A = \frac{x}{b}$, resulting in $x = b\cos A$.
- Substitute $x = b\cos A$ into the equation $a^2 = b^2 + c^2 - 2cx$. What do you notice? ($a^2 = b^2 + c^2 - 2cb\cos A$)

Students can then express the formula in different forms to find the lengths of the other sides of the triangles.

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Journal

- Ask students to explain how the Law of Cosines validates the Pythagorean theorem if the included angle is 90°.

(T3.3)

Resources/Notes

Authorized Resource

Pre-Calculus 11

2.4 The Cosine Law
SB: pp. 114-125
TR: pp. 81-87
BLM: 2-3, 2-8

Trigonometry

Outcomes

Students will be expected to

T3 Continued ...

Achievement Indicator:

T3.6 *Sketch a diagram and solve a problem, using the cosine law.*

Elaborations—Strategies for Learning and Teaching

The cosine law can be used to determine an unknown side or angle measure in a triangle. Continue to encourage students to draw diagrams with both the given and unknown information marked when solving problems. Ask them what information is needed to use the cosine law. They should consider why the cosine law is the only option to find the unknown angle if three sides are known or if two sides and the included angle are known.

When three sides of a triangle are known, students will use the cosine law to find one of the angles. Some students may rearrange the equation to solve for a particular angle. Others may substitute the unknown values into the cosine law and then rearrange the equation to find the angle. It is important for them to recognize they have a choice when trying to find the second angle. They can either use the cosine law or the sine law. Students should notice the third angle can then be determined using the sum of the angles in a triangle.

When solving triangles, encourage students to consider the following questions:

- What is the given information?
- What am I trying to solve for?
- With the given information, should I use the sine law or the cosine law? Is there a choice?
- Which form of the cosine law do I use to solve for an unknown side? Which form do I use to solve for an unknown angle?

If students know two sides and a non-included angle, they can use the cosine law in conjunction with the sine law to find the other side. As an alternative, they could apply the sine law twice. Students have to be exposed to numerous examples to find the method that works best for them.

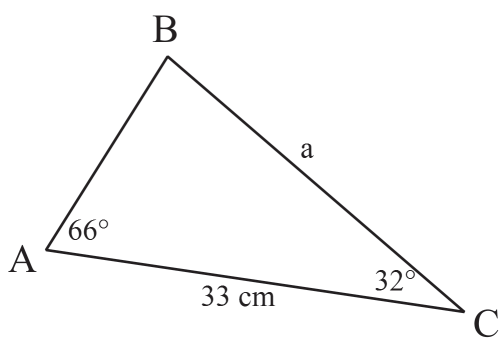
When working with the cosine law, students sometimes incorrectly apply the order of operations. When asked to simplify $a^2 = 365 - 360\cos 70°$, for example, they often write $a^2 = 5\cos 70°$. To avoid this error, teachers should emphasize that multiplication is to be completed before subtraction.

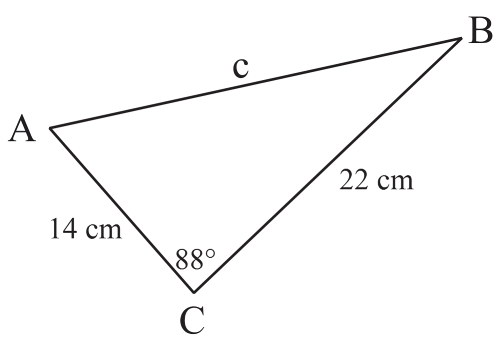
General Outcome: Develop trigonometric reasoning.

Suggested Assessment Strategies

Performance

- In the activity, *Four Corners*, students have to think about which method they would use to solve a triangle. Post four signs, one in each corner labelled sine law, cosine law, Pythagorean theorem, trigonometric ratios. Provide each student with one triangle. Instruct the students to make a decision as to which method they would use to find the missing angle or side and to stand in the corner where it is labelled. Once students are all placed, ask them to discuss why their triangle(s) would be best solved using that particular method. Sample triangles are given below:





(T3.4, T3.6)

Resources/Notes

Authorized Resource

Pre-Calculus 11

2.4 The Cosine Law
SB: pp. 114-125
TR: pp. 81-87
BLM: 2-3, 2-8

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/trig.html

- classroom clip of *Four Corners*
- printable file for *Four Corners* activity

Quadratic Functions

Suggested Time: 13 Hours

Unit Overview

Focus and Context

In this unit, students will be introduced to standard form and vertex form of a quadratic function. They will investigate the characteristics of a quadratic function using features such as x-and y-intercepts, vertex, axis of symmetry, domain and range.

Students will solve problems involving situations that can be modelled by quadratic functions.

Outcomes Framework

GCO

Develop algebraic and graphical reasoning through the study of relations.

SCO RF3

Analyze quadratics of the form
$y = a(x - p)^2 + q$, $a \neq 0$, and determine the:
- vertex
- domain and range
- direction of opening
- axis of symmetry
- x-and y-intercepts.

SCO RF4

Analyze quadratic functions of the form
$y = ax^2 + bx + c$, $a \neq 0$, to identify characteristics of the corresponding graph, including:
- vertex
- domain and range
- direction of opening
- axis of symmetry
- x-and y-intercepts

and to solve problems.

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Algebra and Number	Relations and Functions	Relations and Functions
AN4 Demonstrate an understanding of the multiplication of polynomial expressions (limited to monomials, binomials and trinomials) concretely, pictorially and symbolically. [CN, R, V] AN5 Demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially and symbolically. [C, CN, R, V]	RF3 Analyze quadratics of the form $y = a(x - p)^2 + q$, $a \neq 0$, and determine the: • vertex • domain and range • direction of opening • axis of symmetry • x- and y-intercepts. [CN, R,T,V] RF4 Analyze quadratic functions of the form $y = ax^2 + bx + c$, $a \neq 0$, to identify characteristics of the corresponding graph, including: • vertex • domain and range • direction of opening • axis of symmetry • x-and y-intercepts and to solve problems. [CN, PS, R, T, V]	RF10 Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree ≤ 5 with integral coefficients). [C, CN, ME] RF11 Graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5). [C, CN, T, V]

Mathematical Processes

[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

Relations and Functions

Outcomes

Students will be expected to

RF3 Analyze quadratics of the form $y=a(x-p)^2+q,\ a \neq 0$, **and determine the:**

- vertex
- domain and range
- direction of opening
- axis of symmetry
- x-and y-intercepts.

[CN, R,T, V]

Achievement Indicator:

RF3.1 *Explain why a function given in the form* $y = a(x-p)^2 + q$ *is a quadratic function.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students were introduced to functional notation through work with linear functions. They were introduced to the terms “relation” and “function” and determined if a relation was a function (RF2). They also determined the domain value of a linear function given a range value and vice versa (RF8). The domain and range of a graph was written using interval notation or set notation (RF1). In this unit, students will be introduced to quadratic functions. They will first examine quadratic functions expressed in vertex form, $y = a(x-p)^2 + q$, and determine the x-and y-intercepts, vertex, axis of symmetry, direction of opening, and domain and range. They will also sketch graphs using the characteristics above as well as transformations. Using inductive reasoning, students will study the effect of changing the parameters a, p and q on the shape of the graph and on the number of x-intercepts.

Before students are exposed to the vertex form of a quadratic, they need to become familiar with the shape of a quadratic function and how to identify a quadratic function. The terms “quadratic” and “parabola” are new to students. This will be their first exposure to functions that are non-linear.

Students should have an opportunity to investigate what makes a quadratic function. Ask students to multiply two linear equations or square a binomial of the form $ax + b$, for example, $y = (x + 1)(x - 4)$ and $y = (3x - 2)^2$. Ask students what they notice in terms of the degree of the polynomial and if this is similar to expanding functions of the form $y = a(x-p)^2 + q$.

Projectile motion can be used to explain the path of a baseball or a skier in flight. To help students visualize the motion of a projectile, toss a ball to a student. Ask students to describe the path of the ball to a partner and sketch the path of the height of the ball over time (the independent axis represents time and the dependent axis represents height of the ball). Encourage them to share their graphs with other students. Students should conclude that the graph resembles a U-shape. Ask them to think of other examples that might fit the diagrams of parabolas that open upward or downward. They should explain why they think quadratic relations represent functions.

Characteristics of the resulting parabola, such as the vertex and axis of symmetry, should be discussed. It is important for them to recognize that each point on one side of a parabola has a corresponding point reflected in the axis of symmetry.

It is important to note, however, that obtaining a U-shaped graph does not guarantee the function is quadratic. Other functions may produce a similarly shaped graph. The graph of $y = x^4$, for example, may be misinterpreted as being parabolic if not examined closely.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Provide the following table. Ask students to describe the reasoning they used to decide whether each statement is true or false.

Polynomial Function	Classification	True or False
$y = 5(x + 3)$	Linear	
$y = 5(x^2 + 3)$	Quadratic	
$y = 5^2(x + 3)$	Quadratic	
$y = 5x(x + 3)$	Linear	
$y = (5x + 1)(x + 3)$	Quadratic	
$y = 5(x + 3)^2 + 2$	Quadratic	

(RF3.1)

Resources/Notes

Authorized Resource

Pre-Calculus 11

3.1 Investigating Quadratic Functions in Vertex Form

Student Book (SB): pp. 142-162
Teacher Resource (TR): pp. 103-112
Blackline Master (BLM): 3-3, 3-4

Relations and Functions

Outcomes

Students will be expected to

RF3 Continued ...

Achievement Indicators:

RF3.2 *Compare the graphs of a set of functions of the form $y = ax^2$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of a.*

RF3.3 *Compare the graphs of a set of functions of the form $y = (x - p)^2$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of p.*

RF3.4 *Compare the graphs of a set of functions of the form $y = x^2 + q$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of q.*

Elaborations—Strategies for Learning and Teaching

The characteristics of the quadratic function, $y = a(x - p)^2 + q$, should be developed through an investigation where the parameters a, p and q are manipulated individually. Technology, such as graphing calculators, Desmos™, FX Draw™, Graphmatica™, Geometer's Sketchpad™, Winplot™, Geogebra™, or other suitable graphing software, should be used.

Students will first investigate the effect of changing the value of a by comparing quadratic functions $y = x^2$ and $y = ax^2$. As they compare the graphs of $y = 2x^2$ and $y = -2x^2$ for example, to the graph of $y = x^2$, use prompts such as the following to promote student discussion:

- What happens to the direction of opening of the quadratic if $a < 0$ or $a > 0$?
- Is the shape of the parabola affected by the parameter a?
- Are some graphs wider or narrower when compared to the graph of $y = x^2$?
- What is the impact on the graph if $a = 0$?
- What effect does parameter a have on the vertex?

Encourage students to pay particular attention to the points and how they change. For example, the point (2,4) on the graph of $y = x^2$ changes to the point (2,8) on the graph of $y = 2x^2$.

Similarly, students can compare points on the graphs of $y = \frac{1}{2}x^2$ and $y = -\frac{1}{2}x^2$ to the graph of $y = x^2$.

Use graphing technologies to examine the effects of manipulating the values of p. Students will investigate the effect of changing the value of p by comparing quadratic functions $y = x^2$ and $y = (x - p)^2$. They could describe how the graphs of $y = (x - 3)^2$ and $y = (x + 3)^2$ compare to the graph of $y = x^2$. Ask questions such as:

- How does each graph change when compared to $y = x^2$?
- What effect does varying p have on the vertex?

Similarly, students will examine the effect of manipulating the value of q. They will investigate the effect of changing the value of q by comparing quadratic functions $y = x^2$ and $y = x^2 + q$. They could describe how the graph of $y = x^2 + 2$ and $y = x^2 - 2$ compares to the graph of $y = x^2$. Ask students what effect varying q has on the vertex.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Performance

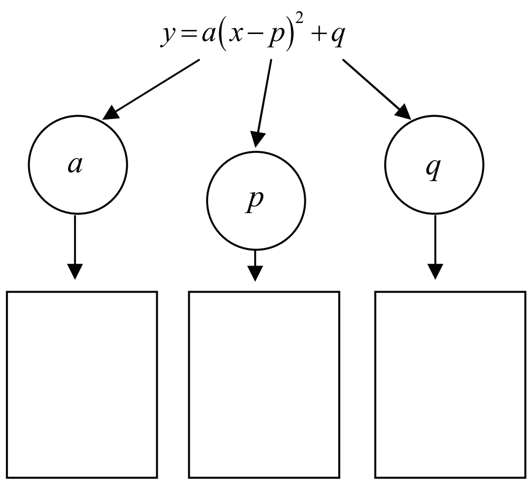
- Ask students to draw a tile from a bag containing various values for the parameter a. They will then proceed to the board or on the wall of a classroom to complete a table, such as the one shown below, indicating the effect of a on the graph of $y = x^2$. Ask students to discuss the results. Be sure to include positive and negative values for a, as well as integers and rational numbers.

Value of a	Function	Opening Up or Down	Narrower or Wider Than $y = x^2$
$a = 2$	$y = 2x^2$	up	narrower
$a = -\frac{1}{2}$	$y = -\frac{1}{2}x^2$	down	wider

(RF3.2)

Paper and Pencil

- Ask students to complete the graphic organizer to describe the effects of the parameters a, p and q on the quadratic function $y = a(x - p)^2 + q$.



(RF3.2, RF3.3, RF3.4)

Interview

- Ask students to explain why $a \neq 0$ when working with quadratics of the form $y = a(x - p)^2 + q$.

(RF3.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11

3.1 Investigating Quadratic Functions in Vertex Form

SB: pp. 142-162
TR: pp. 103-112
BLM: 3-3, 3-4

Relations and Functions

Outcomes

Students will be expected to

RF3 Continued ...

Achievement Indicators:

RF3.5 *Determine the coordinates of the vertex for a quadratic function of the form $y = a(x - p)^2 + q$, and verify with or without technology.*

RF3.6 *Generalize, using inductive reasoning, a rule for determining the coordinates of the vertex for quadratic functions of the form $y = a(x - p)^2 + q$.*

Elaborations—Strategies for Learning and Teaching

In Grade 9, students graphed linear relations using a table of values (9PR2). They also investigated line symmetry (9SS5). In Mathematics 1201, students used technology and/or a table of values to graph linear functions (RF6). They will now extend these concepts to determine the vertex and axis of symmetry of the graph of a quadratic function. When graphing the quadratic function $y = 3(x + 2)^2 - 1$, for example, students can create the following table of values by hand or through the use of graphing technology.

x	-4	-3	-2	-1	0
y	11	2	-1	2	11

Promote student discussion by asking questions such as the following:

- What is the connection between the axis of symmetry of the graph and the vertex?
- What is the equation of the axis of symmetry?
- What is the connection between the maximum or minimum point on the graph and the vertex?
- How can the vertex be obtained directly from $y = a(x - p)^2 + q$?

The goal is for students to recognize that in a table of values, the vertex will be the point with the unique y-value. It is the point at which the graph of the quadratic changes from rising to falling ($a < 0$) or from falling to rising ($a > 0$). That is, it is the maximum or minimum point on the graph. Whether students use a graph or a table of values, they should be able to identify the symmetry of the graph since the points on the parabola that share the same y-coordinate are equidistant from the vertex. Students should also realize that the axis of symmetry of a parabola is the vertical line of symmetry that passes through the vertex and has equation $x = p$.

Since the vertex can be read directly from $y = a(x - p)^2 + q$, this is known as the vertex form of the quadratic function.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Observation

- Ask students to graph $y = 3(x - 2)^2 + 1$ using a table of values or graphing technology. They should compare their graph to the graphs of each relation below and discuss the similarities and differences between the two.

  (i) $y = -3(x - 2)^2 + 1$

  (ii) $y = 3(x - 3)^2 + 1$

  (iii) $y = -\frac{1}{3}(x - 2)^2 + 4$

  (iv) $y = \frac{1}{3}(x - 2)^2 + 1$

  (v) $y = \frac{1}{2}(x + 3)^2 + 2$

  (vi) $y = -2x^2 + 1$

  (vii) $y = 3x^2 + 1$

(RF3.1, RF3.2, RF3.3, RF3.4, RF3.5)

Journal

- Your friend has missed the class on determining the vertex for any quadratic function of the form $y = a(x - p)^2 + q$ and asks you to explain how this is done. Ask students to write a paragraph, with examples, explaining how this is done.

(RF3.6)

Resources/Notes

Authorized Resource

Pre-Calculus 11

3.1 Investigating Quadratic Functions in Vertex Form

SB: pp. 142-162
TR: pp. 103-112
BLM: 3-3, 3-4

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/quad-fun.html

- Visual model of how changes to a, p, and q impact the shape of the graph of the parabola

Relations and Functions

Outcomes

Students will be expected to

RF3 Continued ...
Achievement Indicator:

RF3.7 *Sketch the graph of $y = a(x - p)^2 + q$, using transformations, and identify the vertex, domain and range, direction of opening, axis of symmetry and x- and y-intercepts.*

Elaborations—Strategies for Learning and Teaching

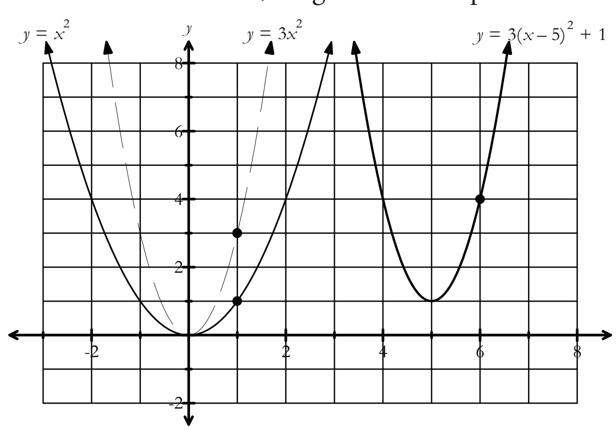
Students previously investigated the effects on the graph of changing the values of a, p, and q in the vertex form of a quadratic function. They also found the vertex directly from the quadratic function.

In Mathematics 1201, students graphed linear relations by plotting the x-and y-intercepts and found the domain and range of various relations (RF8 and RF1). They should recognize that all non-contextual quadratic functions have a domain of $\{x| \; x \in \Re\}$, whereas the range depends on the vertex and the direction of opening. Ask students questions about their observations as they graph and analyze quadratic functions. They should consider the following:

- Why is the domain the set of all real numbers when only some points are plotted from the table of values?
- How is the range related to the direction of opening?

Students should realize that a negative value of a will indicate that the range is less than or equal to q while a positive value of a will indicate that the range is greater than or equal to q.

Students can consolidate their learning by sketching the graph of $y = a(x - p)^2 + q$ using transformations for various values of a, p, and q. They can apply the change in width using the value of a by selecting the vertex and two other points on the graph of $y = x^2$. They can then use the values of p and q to translate the graph. Consider the function $y = 3(x - 5)^2 + 1$. The a value, 3, results in a narrower graph. The points (0, 0), (-1, 1) and (1, 1) on the graph $y = x^2$ will transform to (0, 0), (-1, 3), (1, 3) on the graph of $y = 3x^2$. Students will then translate the graph using the horizontal translation (5 units) and the vertical translation (1 unit). The points transform to (5, 1), (4, 4), and (6, 4). Students should note the vertex and symmetry of the graph and other features such as the domain, range and intercepts.

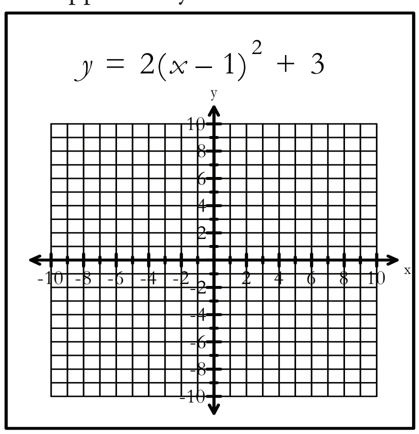


General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

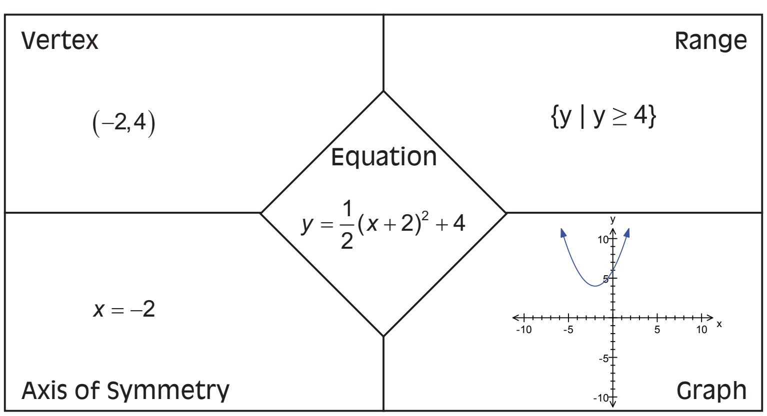
Observation

- Students can work in pairs to construct the graphs of a number of quadratic functions of the form $y = a(x - p)^2 + q$, using transformations for various values of a, p and q. Each group is provided with a handout containing a grid and the function. Place the handout inside a sheet protector so each group can graph a different function using an erasable marker. Ask questions to the group related to the characteristics of the quadratic function (vertex, direction of opening, x-and y-intercepts, domain and range). When completed, ask students to erase the function and pass it along to another group. Repeat the activity, providing students the opportunity to construct a number of graphs.



(RF3.7)

- Students can work in pairs to complete the following quadratic puzzle investigating the characteristics and graphs of various quadratic functions of the form $y = a(x - p)^2 + q$. They should work with 20 puzzle pieces (4 complete puzzles consisting of a function and four related characteristics) to correctly match the characteristics with each function. A sample is shown below.



(RF3.5, RF3.6, RF3.7)

Resources/Notes

Authorized Resource

Pre-Calculus 11

3.1 Investigating Quadratic Functions in Vertex Form

SB: pp. 142-162
TR: pp. 103-112
BLM: 3-3, 3-4

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/quad-fun.html

- *Frayer Model* classroom clip demonstrating students finding the different representations of a quadratic function and completing a puzzle
- *Frayer Model* puzzle pieces

Relations and Functions

Outcomes

Students will be expected to

RF3 Continued...

Achievement Indicators:

RF3.8 *Explain, using examples, how the values of a and q may be used to determine whether a quadratic function has zero, one or two x-intercepts.*

RF3.9 *Write a quadratic function in the form $y = a(x - p)^2 + q$ for a given graph or a set of characteristics of a graph.*

Elaborations—Strategies for Learning and Teaching

Teachers could use a variety of examples to demonstrate the six different combinations of *a* and *q* shown in the table below. The use of graphing technologies or simulation software is recommended here.

Value of *a*	Value of *q*	Number of *x*-intercepts
positive	positive	0
positive	negative	2
positive	0	1
negative	0	1
negative	positive	2
negative	negative	0

Given a graph of a quadratic function, students will determine the quadratic function in the form $y = a(x - p)^2 + q$. They can determine the vertex directly from the graph and hence, the values of *p* and *q*. When determining the value of *a*, students can substitute a point (x, y) that is on the parabola into the quadratic function and solve for *a*.

Sometimes a description of the characteristics of the graph are given rather than the graph. The vertex is given, as a maximum or minimum value, and at least one other point. Using this information, students will determine the quadratic function in vertex form.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Performance

- Ask students to participate in the following card game. Place three headings labelled "No x-intercepts", "One x-intercept", and "Two x-intercepts" on the classroom wall. Make a set of cards for the values of a and q. For example,

  $a = 3, -8, 2.67, \frac{3}{4}$

  $q = 0, -10, 11, \frac{-24}{7}$

  One student will select a card from the "a" deck and a second student will select a card from the "q" deck. The two students then collaborate to determine under which heading the two numbers should be placed. At the end of the activity, they should determine if any of the pairs of cards are misplaced.

(RF3.8)

Paper and Pencil

- Ask students to determine the vertex form of the quadratic function from the given table of values.

x	-6	-5	-4	-3	-2	-1	0
y	1	3	1	-5	-15	-29	-47

(RF3.9)

- Ask students to determine the vertex form of the quadratic function given the following information.
  (i) Range is $\{y \mid y \le 3, y \in \mathbb{R}\}$ and the x-intercepts are -2 and 4.
  (ii) Equation of the axis of symmetry is $x = 2$, the minimum value of y is -5, and the y-intercept is 3.

(RF3.9)

Resources/Notes

Authorized Resource

Pre-Calculus 11

3.1 Investigating Quadratic Functions in Vertex Form

SB: pp. 142-162
TR: pp. 103-112
BLM: 3-3, 3-4

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/quad-fun.html

- *Describe and Draw* classroom clip demonstrating students graphing a quadratic function by listening to the information presented by their partners

Relations and Functions

Outcomes

Students will be expected to

RF4 Analyze quadratic functions of the form $y = ax^2 + bx + c$, $a \neq 0$ to identify characteristics of the corresponding graph, including:

- **vertex**
- **domain and range**
- **direction of opening**
- **axis of symmetry**
- **x-and y-intercepts**

and to solve problems.

[CN, PS, R, T, V]

Achievement Indicators:

RF4.1 *Determine the characteristics of a quadratic function given in the form $y = ax^2 + bx + c$, and explain the strategy used.*

RF4.2 *Sketch the graph of a quadratic function given in the form $y = ax^2 + bx + c$.*

Elaborations—Strategies for Learning and Teaching

In this unit, students will examine quadratic functions expressed in standard form $y = ax^2 + bx + c$ and determine the x- and y-intercepts, vertex (as a maximum or minimum point), axis of symmetry, direction of opening, and domain and range. They will also sketch the graph of a quadratic function using the characteristics above. Students will study the effect of changing the parameters a, b, and c of the equation on the shape of the graph. They will also solve problems involving quadratic functions.

The characteristics of the quadratic function, $y = ax^2 + bx + c,$ should be developed through an investigation where the parameters a, b, and c are manipulated individually. This should be completed through the use of technology. Ask students what parameters affect the width of the graph, the direction of opening, and the y-intercept. Using the graph as a visual, ask students to discuss the following:

- maximum or minimum value
- domain and range
- equation of the axis of symmetry
- x-intercepts

Students should first be exposed to quadratics for which the vertex has integer coordinates to make it easier to identify characteristics precisely.

Students may use the formula $x = -\frac{b}{2a}$ to find the equation of the axis of symmetry for quadratic functions of the form $y = ax^2 + bx + c$. This value can then be used to determine the x-coordinate of the vertex.

$x = -\frac{b}{2a}$ may be derived by completely expanding $y = a(x - p)^2 + q$ and equating the resulting linear coefficient to b, the linear coefficient of $y = ax^2 + bx + c$. Solving for p shows that $p = \frac{-b}{2a}$ which also means the axis of symmetry is $x = -\frac{b}{2a}$.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Performance

- Ask students to work in groups for this activity. Each group should be given a pair of dice (or they can create their own). On the first die, mark two sides each with *a*, *b* and *c*. On the second die, there will be various rational numbers. Students should roll the dice and state the effect on $y = x^2$ of changing the given parameter to the given number.

(RF4.1, RF4.2)

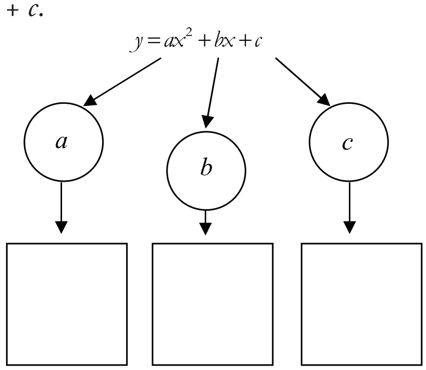
Interview

- Ask students to explain how they could determine whether a quadratic function has either a maximum or minimum value without graphing.

(RF4.1)

Paper and Pencil

- Ask students to complete the following webbing to describe the effects of the parameters *a*, *b* and *c* on the quadratic function $y = ax^2 + bx + c$.



(RF4.1, RF4.2)

- Provide students with a table containing quadratic functions written in standard form and the vertex of each corresponding parabola. Students should complete the table for the values of *a*, *b* and *c* along with the value of $x = -\frac{b}{2a}$.

Function	Vertex	Equation of Axis of Symmetry	a	b	c	$-\frac{b}{2a}$
$y = x^2 - 4x + 7$	(2, 3)					
$y = -2x^2 - 16x - 34$	(-4, -2)					
$y = 3x^2 - 6x + 10$	(1, 7)					

(RF4.1, RF4.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11

3.2 Investigating Quadratic Functions in Standard Form

SB: pp. 163-179
TR: pp. 113-124
BLM: 3-3, 3-5

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/quad-fun.html

- visual model of how changes to *a*, *b*, and *c* impact the shape of the graph of the parabola

Relations and Functions

Outcomes

Students will be expected to

RF4 Continued ...

Achievement Indicators:

RF4.2 *Continued*

RF4.3 *Explain the reasoning for the process of completing the square as shown in a given example.*

Elaborations—Strategies for Learning and Teaching

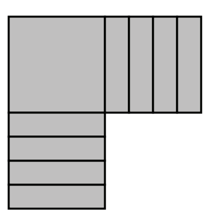
Students should discover that the value of $-\frac{b}{2a}$ is the x-coordinate of the vertex and its connection should be made to the equation of the axis of symmetry. The y-coordinate of the vertex can be found by substituting the x-coordinate into the quadratic function. Note that the y-coordinate of the vertex can also be found using the formula $y=\frac{4ac-b^2}{4a}$. This formula could be developed by substituting $x=-\frac{b}{2a}$ into $y = ax^2 + bx + c$.

In Mathematics 1201, students represented a quadratic expression using algebra tiles. They also demonstrated, through the use of algebra tiles, an understanding of the multiplication of polynomial expressions (limited to monomials, binomials and trinomials), common factoring and trinomial factoring (AN4, AN5).

Students should be given an opportunity to understand the process of creating perfect square trinomials and the patterns formed. The method of completing the square is one method students will use when they rewrite a quadratic equation from standard form to vertex form. The perfect square binomial $(x - p)^2$ is part of the quadratic function in vertex form.

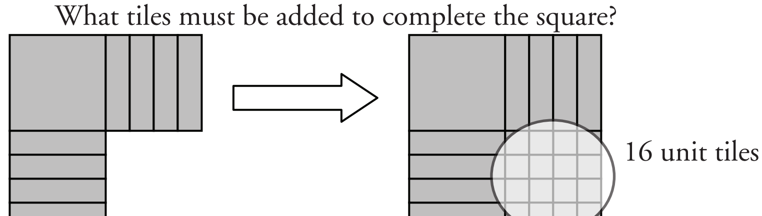
Algebra tiles can be used to visualize how a perfect square trinomial can be formed. Students should first be exposed to examples where $a = 1$ and $c = 0$. Consider the following example:

Ask students to model $x^2 + 8x$. The goal is to find a number c to create a perfect square trinomial $x^2 + 8x + c$. Students will use the algebra tiles to create a square. Ask students why the number of tiles must be split evenly.



Continue to use leading questions, such as the following, to promote discussion:

- What tiles must be added to complete the square?



General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Observation

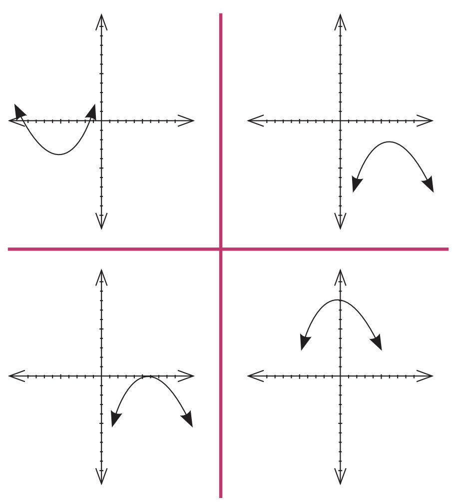
- Ask students to graph the following using a table of values or graphing technology.

  (i) $y = x^2$
  (ii) $y = x^2 + 2$
  (iii) $y = x^2 + 5$
  (iv) $y = x^2 - 3$
  (v) $y = x^2 - 4$

  As teachers observe students' work, ask them to explain the changes in the parabola as the parameter c is manipulated.

(RF4.1, RF4.2)

- Invite students to play WODB (Which One Doesn't Belong?). Show students the following graphs. Each graph could be the one which doesn't belong, but for a different reason. Observe students' reasoning for misconceptions.



(RF4.1, RF4.2)

Interview

- Using algebra tiles, ask students to model and explain the process of completing the square. The teacher photographs the group as they participate in the activity. After a few classes, give students their photograph and ask them to describe what they were doing in the picture. They should write about the activity under the photograph, describing what they were doing and what they learned as a result.

(RF4.3)

Resources/Notes

Authorized Resource

Pre-Calculus 11

3.2 Investigating Quadratic Functions in Standard Form

SB: pp. 163-179
TR: pp. 113-124
BLM: 3-3, 3-5

3.3 Completing the Square

SB: pp. 180-197
TR: pp. 125-133
BLM: 3-3, 3-6

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/quad-fun.html

- graphs for WODB

Relations and Functions

Outcomes

Students will be expected to

RF4 Continued ...

Achievement Indicators:

RF4.3 *Continued*

RF4.4 *Write a quadratic function given in the form $y = ax^2 + bx + c$ as a quadratic function in the form $y = a(x - p)^2 + q$ by completing the square.*

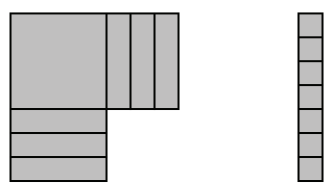
Elaborations—Strategies for Learning and Teaching

- What is the expression that represents the new completed square?
- What is the relationship between the coefficient of the linear term and the constant term?
- What is the trinomial written as the square of a binomial?

Students should realize 16 unit tiles have been added. Since the side length of the square is represented by $x + 4$, the area is $(x + 4)(x + 4)$. This perfect square trinomial $x^2 + 8x + 16$, can be rewritten as $(x + 4)^2$.

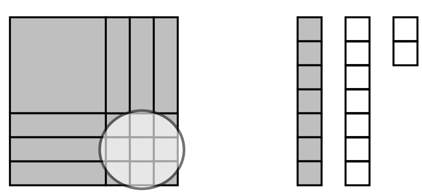
Continue to work with various examples, such as $x^2 + 2x$, and $x^2 + 6x$, to give students an opportunity to describe the pattern that exists. The algebra tile method illustrates the constant term is half the coefficient of the linear term squared.

This would be an opportunity for students to extend their work with algebra tiles to convert an equation from standard form to vertex form using the process of completing the square. They will then convert between the two forms algebraically. Use algebra tiles to model the function $y = x^2 + 6x + 7$. Students will continue to split the tiles evenly and add tiles to form a square leaving the constant term alone.



$x^2 + 6x$ 7 unit tiles

It is important for students to notice that by adding nine extra tiles, the quadratic function has changed. Therefore, students will need to add 9 opposite tiles to balance the function.



$x^2 + 6x + 9$ $7 - 9$

The function is $y = (x^2 + 6x + 9) + (7 - 9)$. They should recognize when a number is added to form a perfect square, its opposite value is also added to keep the original expression unchanged. Students can then rewrite this as $y = (x + 3)^2 - 2$. This visual representation allows students to observe patterns and then record their work symbolically.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

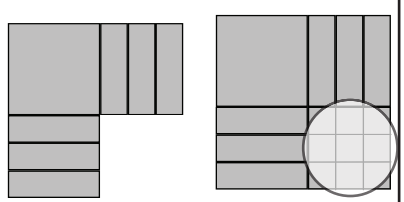
Suggested Assessment Strategies

Paper and Pencil

- Ask students to complete the square for $y = ax^2 + bx + c$ to show that the coordinates of the vertex are $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.

(RF4.1, RF4.3)

- Ask students to complete a table investigating the relationship between the value of b and c when completing the square. They should share their findings with the class. A sample is shown below.

Binomial of the form $x^2 + bx$	Modeled with algera tiles	Perfect square
$x^2 + 6x$		$x^2 + 6x + 9$

What was added to complete the square?	Factored and expanded form	Value of coefficient c	How are b and c related?
9	$(x + 3)(x + 3) = x^2 + 6x + 9$	9	$c = \left(\frac{1}{2}b\right)^2$

(RF4.3)

Resources/Notes

Authorized Resource

Pre-Calculus 11

3.3 Completing the Square

SB: pp. 180-197
TR: pp. 125-133
BLM: 3-3, 3-6

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/quad-fun.html

- using algebra tiles to model completing the square when $a > 1$

Relations and Functions

Outcomes

Students will be expected to

RF4 Continued ...

Achievement Indicator:

RF4.4 *Continued*

Elaborations—Strategies for Learning and Teaching

Ask students why it is to their advantage to write an equation in vertex form. Students should emphasize that the coordinates of the vertex (p, q) can be directly determined when written in this form. Hence, the maximum or minimum point is known. Students should first be exposed to quadratic functions where the leading coefficient is 1 and then progress to examples where $a \neq 1$.

Algebraically, completing the square parallels the algebra tile approach. Using the same function $y = x^2 + 6x + 7$, students will first group the quadratic and linear terms:

$y = (x^2 + 6x + \_\_) + 7\ \_\_$

From the work with algebra tiles, students know that they need to add nine tiles to "complete the square", and subtract nine tiles to keep the original equation unchanged.

$y = (x^2 + 6x + \underline{\ 9\ }) + 7\ \underline{-9}$

$y = (x + 3)^2 - 2$

For quadratics with $a \neq 1$, completing the square algebraically is more robust. For example,

$$y=-2x^2+10x-6$$
$$y=-2(x^2-5x)-6$$
$$y=-2\left(x^2-5x+\frac{25}{4}\right)-6+\frac{25}{2}$$
$$y=-2\left(x-\frac{5}{2}\right)^2-6+\frac{25}{2}$$
$$y=-2\left(x-\frac{5}{2}\right)^2+\frac{13}{2}$$

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Performance

- Ask students to work in pairs for this activity. Give each pair of students a quadratic equation in standard form and ask them to rewrite the equation in vertex form. Ask one student to write the first line of the solution and then pass it to the second student. The second student will verify the workings to determine if an error is present. If there is an error present, the student will correct it and then write the second line of the solution and pass it along to their partner. This process continues until the solution is complete.

(RF4.4, RF4.5)

- Ask students to use algebra tiles to write each in vertex form:

(i) $y = x^2 + 8x + 11$

(ii) $y = x^2 - 8x + 11$

(RF4.4)

- Ask students to convert the following to vertex form algebraically:

(i) $y = 2x^2 + x + 1$

(ii) $y = -5x^2 - 15x - 12$

(RF4.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11

3.3 Completing the Square

SB: pp. 180-197
TR: pp. 125-133
BLM: 3-3, 3-6

Relations and Functions

Outcomes

Students will be expected to

RF4 Continued ...

Achievement Indicators:

RF4.5 *Identify, explain and correct errors in an example of completing the square.*

RF4.6 *Verify, with or without technology, that a quadratic function in the form* $y = ax^2 + bx + c$ *represents the same function as a given quadratic function in the form* $y = a(x - p)^2 + q$.

Elaborations—Strategies for Learning and Teaching

Common errors occur when converting a quadratic function from standard form to vertex form. A quadratic function where $a \neq 1$, for example, sometimes causes difficulty for students. Ask students to consider the following example and discuss with them the possible errors that may occur:

$$y = -3x^2 + 18x - 23$$
$$y = -3(x^2 - 6x) - 23$$
$$y = -3(x^2 - 6x + 9) - 23 + 27$$
$$y = -3(x - 3)^2 + 4$$

- The common factor (-3) might not be factored out from both the quadratic and linear terms.
- There may be an incorrect sign on the linear term when a negative leading coefficient is factored out.
- The constant term inside the parentheses could be doubled instead of squared.
- When a perfect square is created, the constant term inside the parentheses may not be multiplied by the common factor to produce the compensated term.
- The perfect square trinomial may be incorrectly factored.

When converting a quadratic function from vertex form to standard form, students will expand the perfect square trinomial and then combine like terms. Students should also verify that a quadratic function in standard form represents the same function in vertex form. They can use the method of completing the square to compare the functions or compare the graphs of both functions. Ask students what features of the graph must be the same.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- In groups of two, ask students to play the *Domino Game*. Provide students with 10 domino cards. One side of the card contains a quadratic function in standard form, while the other side contains a quadratic function in vertex form. Ask each group to lay the dominos out such that the standard form on one card will match with the correct vertex form on another. They will eventually form a complete loop, with the first card matching the last card. Some sample cards are shown below:

$y = (x+5)^2 - 3$	$y = 2x^2 - 12x + 23$
$y = 2(x-3)^2 + 5$	$y = -x^2 - 2x + 2$
$y = -(x+1)^2 + 3$	$y = x^2 + 10x + 22$

(RF4.6)

- In groups of two, ask students to move around the classroom to various stations where solutions have been posted outlining the process of completing the square. At each station the solution could contain one or more errors that the group has to identify and then produce correct solutions for each of the problems.

(RF4.5)

Resources/Notes

Authorized Resource

Pre-Calculus 11

3.3 Completing the Square

SB: pp. 180-200
TR: pp. 125-133
BLM: 3-3, 3-6

Relations and Functions

Outcomes

Students will be expected to

RF4 Continued ...

Achievement Indicators:

RF4.7 *Write a quadratic function that models a given situation, and explain any assumptions made.*

RF4.8 *Solve a problem, with or without technology, by analyzing a quadratic function.*

Elaborations—Strategies for Learning and Teaching

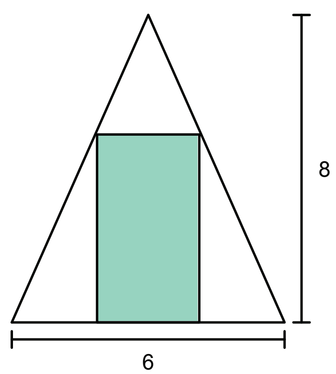
Students will be expected to write a quadratic function that models a situation and then solve the problem. They will be exposed to problems where they will determine the maximum or minimum value. Contextual problems could involve maximum revenue, finding the maximum possible area, the maximum height, or a minimum product. It is important for students to understand the terminology that is being used in the problems. They should realize that the maximum or minimum value is the y-value of the vertex of the function.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to answer the following :
  - (i) A stream of water from a fountain forms a parabolic shape. Given the spout on the fountain is 5 cm high and the maximum height reached by the water is 14 cm at a distance of 6 cm from the spout, what is the height of the water when it is 8 cm from the spout?
  - (ii) A student makes and sells necklaces at the beach during the summer months. The material for each necklace costs \$6.00 with sales of about 20 per day at \$10.00 each. A survey shows that for every dollar increase in the price, sales drop by two necklaces a day. What price should be set for the necklaces to maximize profit?
  - (iii) Find the dimensions of the rectangle of maximum area that can be inscribed in an isosceles triangle of altitude 8 and base 6. (Hint: Use similar triangles to express the height of the rectangle in terms of its base).



(RF4.7, RF4.8)

Resources/Notes

Authorized Resource

Pre-Calculus 11

3.3 Completing the Square

SB: pp. 180-197
TR: pp. 125-133
BLM: 3-3, 3-6

Quadratic Equations

Suggested Time: 16 Hours

Unit Overview

Focus and Context

In this unit, students will extend their factoring skills and use a variety of strategies to determine the roots of quadratic equations. They will make the connection that the *x*-intercepts of the graph or the zeros of the quadratic function correspond to the solutions, or roots, of the quadratic equation.

Students will explore various algebraic methods for solving quadratic equations including factoring, completing the square and the quadratic formula. They will use the discriminant to determine the number and type of roots from quadratic equations.

Contextual problems will also be solved by modelling a situation with a quadratic equation.

Outcomes Framework

GCO

Develop algebraic and graphical reasoning through the study of relations.

SCO RF1

Factor polynomial expressions of the form:

- $ax^2 + bx + c,\ a \neq 0$
- $a^2x^2 - b^2y^2,\ a \neq 0,\ b \neq 0$
- $a(f(x))^2 + b(f(x)) + c,\ a \neq 0$
- $a^2(f(x))^2 - b^2(g(y))^2,\ a \neq 0,\ b \neq 0$

where a, b and c are rational numbers.

SCO RF5

Solve problems that involve quadratic equations.

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Algebra and Number	Relations and Functions	Relations and Functions
AN5 Demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially and symbolically. [C, CN, R, V]	RF1 Factor polynomial expressions of the form: • $ax^2 + bx + c, a \neq 0$ • $a^2x^2 - b^2y^2, a \neq 0, b \neq 0$ • $a(f(x))^2 + b(f(x)) + c, a \neq 0$ • $a^2(f(x))^2 - b^2(g(y))^2, a \neq 0, b \neq 0$ where a, b and c are rational numbers. [CN, ME, R] RF5 Solve problems that involve quadratic equations. [C, CN, PS, R, T, V]	RF9 Solve problems that involve exponential and logarithmic equations. [C, CN, PS, R] RF10 Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree ≤ 5 with integral coefficients). [C, CN, ME]

Mathematical Processes

[**C**]	Communication	[**PS**]	Problem Solving
[**CN**]	Connections	[**R**]	Reasoning
[**ME**]	Mental Mathematics and Estimation	[**T**]	Technology
		[**V**]	Visualization

Relations and Functions

Outcomes

Students will be expected to

RF5 Solve problems that involve quadratic equations.
[C, CN, PS, R, T, V]

Achievement Indicators:

RF5.1 *Explain, using examples, the relationship among the roots of a quadratic equation, the zeros of the corresponding quadratic function and the x-intercepts of the graph of the quadratic function.*

RF5.2 *Solve a quadratic equation of the form $ax^2 + bx + c = 0$ by using strategies such as:*
- *determining square roots*
- *factoring*
- *completing the square*
- *applying the quadratic formula*
- *graphing its corresponding function.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students factored differences of squares, perfect square trinomials and polynomials of the form $x^2 + bx + c$ and $ax^2 + bx + c$ (AN5).

In the previous unit, students were introduced to quadratic functions expressed in standard form and vertex form. They sketched graphs using characteristics such as x-and y-intercepts, vertex, axis of symmetry and domain and range. Students also converted a quadratic function in standard form to vertex form using the process of completing the square.

In this unit, students will solve quadratic equations using various strategies, including factoring, completing the square, applying the quadratic formula or using a graph.

Students have been exposed to the graph of a quadratic function and the points where a parabola crosses the x-axis. They are aware a quadratic function can have zero, one or two x-intercepts. When solving a quadratic equation of the form $ax^2 + bx + c = 0$, students can graph the corresponding quadratic function and determine the x-intercepts. They can use a table of values or graphing technology to make the connection between the x-intercepts of the graph and the roots of the quadratic equation.

It is important for students to distinguish between the terms roots, zeros and x-intercepts, and to use the correct term in a given situation. The x-intercepts of the graph or the zeros of the quadratic function correspond to the roots of the quadratic equation. Students could be asked to find the roots of the equation $x^2 - 7x + 12 = 0$, find the zeros of $f(x) = x^2 - 7x + 12$, or determine the x-intercepts of $y = x^2 - 7x + 12$. In each case they are solving $x^2 - 7x + 12 = 0$ and arriving at the solution $x = 3$ or $x = 4$.

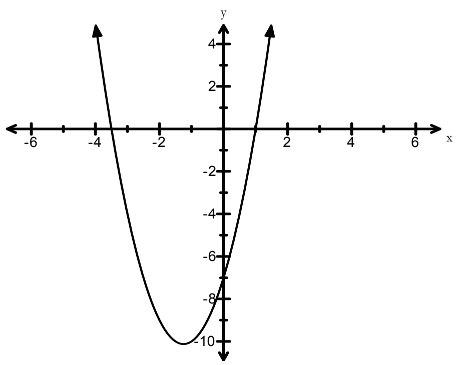
Discuss with students what it means to solve a quadratic problem graphically and the limitations of this method. They will then work with quadratics algebraically and decide when it is best to factor, complete the square or use the quadratic formula.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to answer the following:
  - (i) Find the zeros of $f(x) = 2x^2 + 5x - 7$.
  - (ii) Identify the x-intercepts of the graph.



  - (iii) Find the roots of $2x^2 + 5x - 7= 0$.
  - (iv) What do you notice about the answers to the above questions?

(RF5.1, RF5.2)

- Invite students to play WODB (Which One Doesn't Belong?). Show students the following equations. Each equation could be the one which doesn't belong, but for a different reason. Observe students' reasoning for misconceptions.

$y = t^2 + 6t + 5$	$y = (2x + 1)(x + 5)$
$y = (x^2 - 4)(x^2 - 25)$	$y = x^2 - x - 12$

(RF5.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11

4.1 Graphical Solutions of Quadratic Equations

Student Book (SB): pp. 206-217
Teacher Resource (TR): pp. 143-148
Blackline Master (BLM): 4-3, 4-4

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/quad-eqns.html

- WODB equations

Relations and Functions

Outcomes

Students will be expected to

RF1 Factor polynomial expressions of the form:
$ax^2 + bx + c, a \neq 0$

- $a^2x^2 - b^2y^2, a \neq 0, b \neq 0$
- $a(f(x))^2 + b(f(x)) + c, a \neq 0$
- $a^2(f(x))^2 - b^2(g(y))^2, a \neq 0, b \neq 0$

where a, b and c are rational numbers.
[CN, ME, R]

Achievement Indicators:

RF1.1 *Factor a given polynomial expression that requires the identification of common factors.*

RF1.2 *Factor a given polynomial expression of the form:*

- $ax^2 + bx + c, a \neq 0$
- $a^2x^2 - b^2y^2, a \neq 0, b \neq 0$

RF1.3 *Determine whether a given binomial is a factor for a given polynomial expression, and explain why or why not.*

RF1.4 *Factor a given polynomial expression that has a quadratic pattern, including:*

- $a(f(x))^2+b(f(x))+c, a \neq 0$
- $a^2(f(x))^2-b^2(g(y))^2, a \neq 0, b \neq 0$

Elaborations—Strategies for Learning and Teaching

Students were first exposed to factoring polynomials in Mathematics 1201 (AN5). They removed the greatest common factor from the terms of a polynomial and factored polynomials of the form $x^2 + bx + c$ and $ax^2 + bx + c$. As well, students factored perfect square trinomials, difference of squares and trinomials in two variables.

In this unit, students will extend their knowledge of factoring trinomials and differences of squares to factoring polynomials of the form $a^2x^2 - b^2y^2, a \neq 0, b \neq 0, a(f(x))^2 + b(f(x)) + c, a \neq 0$ and $a^2(f(x))^2 - b^2(g(y))^2, a \neq 0, b \neq 0$.

In Mathematics 1201, students worked extensively with factoring polynomial expressions. They were introduced to factoring using concrete and pictorial models and then moved to a symbolic representation. Most of the previous work dealt with integer coefficients. Remind students of the strategies used when factoring polynomial expressions, including the removal of the greatest common factor and the method of decomposition (AN5). In this course, students are expected to be proficient with factoring at a symbolic level, including expressions with rational coefficients. Students should also be given the opportunity to apply their own personal strategies.

Students should recognize when factoring expressions such as $x^2 + 6x + 8$, possible binomial factors would only contain factors of 8. Therefore, a student should realize that $x + 5$, for example, would not be a possible factor of $x^2 + 6x + 8$. This will improve their factoring skills which will be useful later when using factoring to solve a given quadratic equation.

Factoring trinomials and differences of squares will be extended to factoring polynomials of the form $a(f(x))^2 + b(f(x)) + c, a \neq 0$ and $a^2(f(x))^2 - b^2(g(y))^2, a \neq 0, b \neq 0$. Students can attempt these problems using different methods. Using a substitution, they can treat the embedded functions as a single variable and continue to factor using their previous skills, and then use substitution to complete the factoring. Alternatively, for expressions in one variable, they can expand and group like terms and then factor.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Performance

- For the game *Three in a Row*, students are given a 5 × 5 game board which consists of 25 squares. Each square will have a quadratic equation written on it. Ask them to play in pairs and take turns solving each quadratic equation with the aim to be the first person to make three in a row.

(RF1.1, RF1.2)

Paper and Pencil

- Ask students to determine two values of n that will allow the polynomial $25b^2 + nb + 49$ to be a perfect square trinomial. They should use them both to factor the trinomial.

(RF1.2)

- Ask students to list possible binomial factors if the following expression $x^2 + bx + 24$ can be factored.

(RF1.3)

Interview

- Ask students to respond to the following:
  (i) Explain why $9x^2 – 16y^2$ can be factored but $9x^2 + 16y^2$ cannot be factored.

  (ii) Explain why $(2x - 3y)^2 \neq 4x^2 - 9y^2$.

(RF1.2)

- When Delia factored $x^2 - x - 12$, she said one of her binomial factors was $x + 5$. Without actually factoring the trinomial, ask students to explain why her response is incorrect.

(RF1.3)

Resources/Notes

Authorized Resource

Pre-Calculus 11

4.2 Factoring Quadratic Equations

SB: pp. 218-233
TR: pp. 149-154
BLM: 4-3, 4-5

Relations and Functions

Outcomes

Students will be expected to

RF1, RF5 Continued ...
Achievement Indicators:

RF1.4 *Continued*

RF5.2 *Continued*

Elaborations—Strategies for Learning and Teaching

Consider the following example to compare the two methods of factoring these expressions in one variable:

Factor: $2(x - 2)^2 + 7(x - 2) + 5$

Substitute $p = (x - 2)$	Expand $(x - 2)^2$
$2p^2 + 7p + 5$	$2(x-2)(x-2)+7(x-2)+5$
$(2p + 5)(p + 1)$	$2(x^2 - 4x + 4) + 7(x - 2) + 5$
$[2(x - 2) + 5][(x - 2) + 1]$	$2x^2 - 8x + 8 + 7x - 14 + 5$
$(2x - 4 + 5)(x - 2 + 1)$	$2x^2 -1x - 1$
$(2x + 1)(x - 1)$	$(2x + 1)(x - 1)$

Regardless of the method used, remind students that it is important to use brackets for all substitution.

Students solved a quadratic equation graphically by finding the x-intercepts. They will now solve using algebraic methods. They will explore different strategies to find values for the variable that make the function equal to zero.

Students can use factoring to solve equations. Once a quadratic equation has been factored, use the zero product property to determine the roots. If students were asked to solve $5x^2 + 14x - 3 = 0$, for example, $5x - 1 = 0$ or $x + 3 = 0$. Remind students to substitute the value of each root into the original equation to verify that the value makes the equation true.

Provide students with examples where a quadratic equation of the form $ax^2 + bx + c = 0$ is missing the b or c value. Consider a quadratic equation where $c = 0$. When solving $x^2 - 5x = 0$, for example, students might factor $x(x - 5) = 0$ and then solve by dividing both sides of the equation by x, resulting in only one solution. Ask students if this is correct and whether a root has been eliminated. Encourage students to verify their answers by using the zero product property or by substituting their solutions into the original equation.

Expose students to quadratic equations similar to $x^2 - 8 = 0$ or $(x - 2)^2 - 49 = 0$. They can isolate the squared term and take the square root of both sides of the equation. Remind students there are two possible solutions to these equations. They should also be asked to solve equations in which the right hand side of the equation does not equal 0 (i.e., $3x^2 - 4 = x$).

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Interview

- Ask students to create an example of a quadratic function that cannot be factored. They should explain their reasoning.

(RF1.1, RF1.2, RF1.3, RF1.4)

- After factoring the different types of polynomials, ask students which ones they found easiest to factor and why. Which ones were most difficult and why?

(RF1.1, RF1.2, RF1.3, RF1.4)

Paper and Pencil

- Ask students to factor $4(x + 2)^2 - 9(y + 1)^2$ using two methods. They should explain which method they prefer and why.

(RF1.4)

Performance

- Two sets of different coloured cards are required for this activity. One set will contain quadratic equations and the other set will have their corresponding solutions. Ask students to lay out the cards and match the equation card with its corresponding solution card.

$x^2-2x-24=0$	$x=\frac{5}{2};\ x=2$	$2x^2+5x+3=0$	$x=\frac{3}{2};\ x=-1$
$2x^2-5x+3=0$	$2x^2+7x+5=0$	$x=-\frac{3}{2};\ x=1$	$x=4;\ x=-6$
$x^2+2x-24=0$	$2x^2-9x+10=0$	$2x^2-x-3=0$	$x=-4;\ x=6$
$2x^2+x-3=0$	$x=-\frac{5}{2};x=-1$	$x=\frac{3}{2};x=1$	$x=-\frac{3}{2};x=-1$

(RF5.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11

4.2 Factoring Quadratic Equations

SB: pp. 218-233
TR: pp. 149-154
BLM: 4-3, 4-5

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/quad-eqns.html

- interactive quadratic solving and graphing site

Relations and Functions

Outcomes

Students will be expected to

RF5 Continued ...

Achievement Indicators:

RF5.2 *Continued*

RF5.3 *Derive the quadratic formula, using deductive reasoning.*

Elaborations—Strategies for Learning and Teaching

When students are exposed to quadratic equations that cannot be factored, they have the option to complete the square or apply the quadratic formula. The process of completing the square was introduced in the previous unit when students converted a quadratic function from standard form to vertex form. They will now use completing the square to determine the roots of a quadratic equation in standard form. Once students solve a quadratic equation using this method, provide students the graph of the corresponding function to reinforce the connection between the roots of the equation and the x-intercepts of the graph.

It is important for students to understand how the quadratic formula is developed before they apply it to quadratic equations in standard form. They should use a numerical example before moving to the general form $ax^2 + bx + c = 0$. Ask them to complete the square using an example similar to $3x^2 - 7x + 1 = 0$. Assist students as they follow the same procedure to derive the quadratic formula for $ax^2 + bx + c = 0$. Once the quadratic formula has been derived it can be used to find the roots of any quadratic equation in standard form.

$$
\begin{aligned}
&3x^2 - 7x + 1 = 0\\
&3(x^2 - \tfrac{7}{3}x) + 1 = 0\\
&3(x^2 - \tfrac{7}{3}x + \tfrac{49}{36}) - \tfrac{49}{12} + 1 = 0\\
&3(x - \tfrac{7}{6})^2 = \tfrac{49}{12} - 1\\
&3(x - \tfrac{7}{6})^2 = \tfrac{49}{12} - \tfrac{12}{12}\\
&3(x - \tfrac{7}{6})^2 = \tfrac{37}{12}\\
&(x - \tfrac{7}{6})^2 = \tfrac{37}{36}\\
&x - \tfrac{7}{6} = \pm\tfrac{\sqrt{37}}{6}\\
&x = \tfrac{7}{6} \pm \tfrac{\sqrt{37}}{6}\\
&x = \tfrac{7 \pm \sqrt{37}}{6}
\end{aligned}
$$

$$
\begin{aligned}
&ax^2 + bx + c = 0\\
&a(x^2 + \tfrac{b}{a}x) + c = 0\\
&a(x^2 + \tfrac{b}{a}x + \tfrac{b^2}{4a^2}) - \tfrac{ab^2}{4a^2} + c = 0\\
&a(x + \tfrac{b}{2a})^2 - \tfrac{b^2}{4a} + c = 0\\
&a(x + \tfrac{b}{2a})^2 - \tfrac{b^2}{4a} + \tfrac{4ac}{4a} = 0\\
&a(x + \tfrac{b}{2a})^2 = \tfrac{b^2 - 4ac}{4a}\\
&(x + \tfrac{b}{2a})^2 = \tfrac{b^2 - 4ac}{4a^2}\\
&x + \tfrac{b}{2a} = \pm\sqrt{\tfrac{b^2 - 4ac}{4a^2}}\\
&x + \tfrac{b}{2a} = \pm\tfrac{\sqrt{b^2 - 4ac}}{2a}\\
&x = -\tfrac{b}{2a} \pm \tfrac{\sqrt{b^2 - 4ac}}{2a}\\
&x = \tfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

Students may need assistance when adding rationals ($-\frac{b^2}{4a} + c$) and simplifying variable roots ($\sqrt{4a^2}$). These topics are covered in greater detail later in this course in the Radical Expressions Unit and the Rational Expressions Unit.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Performance

- For the game *Quadratic Bingo*, a list of approximately 50 possible sets of solutions to various quadratic equations will be placed on the board. Each student is given a blank bingo card. They will randomly fill in the 24 squares of their card with one set of solutions. They are now ready to play BINGO. Present one quadratic equation at a time for students to solve. If the solutions are present on their card, the square is covered. Continue to provide equations to be solved until a student has completed a diagonal, horizontal or vertical line on the card.

Quadratic Equation Bingo!				
$x = 0$ $x = -5$	$x = 6$ $x = 8$	$x = 4$ $x = \frac{1}{2}$	$x = -\frac{1}{3}$ $x = 4$	
		FREE		

(RF5.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11

4.3 Solving Quadratic Equations by Completing the Square

SB: pp. 234-243
TR: pp. 155-159
BLM: 4-3, 4-6

4.4 The Quadratic Formula

SB: pp. 244-257
TR: pp. 160-165
BLM: 4-3, 4-7

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/quad-eqns.html

- Quadratic Formula classroom clip demonstrating students taking turns solving a quadratic equation using the quadratic formula
- Quadratic Bingo

Relations and Functions

Outcomes

Students will be expected to

RF5 Continued...

Achievement Indicators:

RF5.4 *Identify and correct errors in a solution to a quadratic equation.*

RF5.5 *Select a method for solving a quadratic equation, justify the choice, and verify the solution.*

RF5.6 *Explain, using examples, how the discriminant may be used to determine whether a quadratic equation has two, one, or no real (i.e. imaginary) roots; and relate the number of zeros to the graph of the corresponding quadratic function.*

Elaborations—Strategies for Learning and Teaching

It is beneficial to have students analyze solutions that contain errors. Students should be provided with worked solutions of quadratic equations that may or may not contain errors. If errors are present, students should identify the error and provide the correct solution including why/how the error occurred. This reinforces the importance of recording solution steps rather than only giving a final answer.

Some common errors occur when students are simplifying the quadratic formula. These include:

- Apply the quadratic formula without ensuring the equation is written in standard form.
- Use $x = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$ rather than the correct form of the quadratic formula.
- Incorrectly produce two possible common errors if the b value is negative.
  - (i) If b = -2 then $-b = -(-2) = -2$
  - (ii) If b = -2 then $b^2 = -2^2 = -4$
- Incorrectly simplify when applying the quadratic formula.
  - (i) $\frac{8\pm\sqrt{5}}{2} = 4 \pm \sqrt{5}$
  - (ii) $\frac{2+4\sqrt{5}}{2} = \pm 2\sqrt{5}$
- Do not recognize that the ± results in two solutions. Suggest that students work through the solutions separately, showing calculations for both the positive solution and the negative solution.

As students work through the various strategies for solving quadratic equations, they should realize that sometimes one method is more efficient than another. The method students choose to solve a quadratic equation will depend on the way the equation is presented.

Students have solved quadratic equations graphically and algebraically. Regardless of their strategy to solve the equation, they should be able to connect the fact that their algebraic solution is the same as the x-intercepts of the graph. In the previous unit, when functions were written in the form $y = a(x - p)^2 + q$, students investigated how the values of a and q affected the number of x-intercepts. They will now explore how these situations arise when solving equations using the quadratic formula.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to identify the most efficient strategy (i.e., determining square roots, factoring, completing the square, applying the quadratic formula or graphing its corresponding function) when solving each equation. They should justify their choice.

  (i) $(x + 2)^2 - 1 = 15$
  (ii) $49x^2 - 64 = 0$
  (iii) $3x^2 - 11x + 6 = 0$
  (iv) $x^2 - 10x - 24 = 0$
  (v) $3x^2 + 8x + 7 = 0$

(RF5.5)

- Given the quadratic equation $ax^2 + bx + c = 0$, ask students what relationship must be true for the coefficients a, b, and c so that the equation has:

  (i) two distinct real roots
  (ii) two equal real roots
  (iii) no real (imaginary) roots

(RF5.6)

Observation

- Mary, David, and Ron are students in a group. They are given the equation A = $x^2 + 3x - 110$ where A represents the area of a field and x represents the width in metres. The students were asked to find the width if the area was 100 m^2. Each student decided to solve the equation using their own preferred method. Here are their solutions:

Mary

$$x^2 + 3x - 110 = 100$$
$$x^2 + 3x - 210 = 0$$
$$x = \frac{3 \pm \sqrt{9 - (4)(1)(-210)}}{2}$$
$$x = \frac{3 \pm \sqrt{-831}}{2}$$
$$x = \frac{3 \pm 28.827}{2}$$
$$x = 15.9 \text{ or } x = -12.9$$
width is 15.9 m

David

$$x^2 + 3x - 110 = 100$$
$$x^2 + 3x - 10 = 0$$
$$(x + 5)(x - 3) = 0$$
$$x = -5 \text{ or } x = 3$$
width is 3 m

Ron

$$x^2 + 3x - 110 = 0$$
$$x = \frac{-3 \pm \sqrt{9 - (4)(1)(-110)}}{2}$$
$$x = \frac{-3 \pm \sqrt{9 + 440}}{2}$$
$$x = \frac{-3 \pm \sqrt{449}}{2}$$
$$x = \frac{-3 \pm 21.2}{2}$$
$$x = 9.1 \text{ or } x = -12.1$$
width is 9.1 m

Ask students to identify and explain any errors in the students' work. They should then proceed to write the correct solution.

(RF5.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11

4.4 The Quadratic Formula

SB: pp. 244-257
TR: pp. 160-165
BLM: 4-3, 4-7

Relations and Functions

Outcomes

Students will be expected to

RF5 Continued...

Achievement Indicator:

RF5.6 *Continued*

Elaborations—Strategies for Learning and Teaching

When students use the quadratic formula to find roots of a quadratic equation they will be exposed to trying to find the square roots of non-perfect squares and negative numbers. It is important for students to distinguish between exact solutions and approximate solutions.

While students need to be aware that there is no real solution to the square root of a negative number, they should be informally introduced to the complex number system. Students should be introduced to the imaginary unit i, the basis of the imaginary numbers. The following is true of i:

$$i = \sqrt{-1}$$
$$i^2 = -1$$

Students need to know the imaginary unit is defined as a square root of a negative real number. For example, $\sqrt{-9}$ is an imaginary number. Since the imaginary unit is defined as $\sqrt{-1}$, it is noted that $\sqrt{-9} = \sqrt{9(-1)} = \sqrt{9}\sqrt{-1} = 3i$. This will give students an opportunity to represent non-real (imaginary) roots of quadratic equations as complex numbers of the form $a \pm bi$ where a and b are real numbers. No further extensions of complex numbers are necessary within this course.

Students should discuss the conditions that are necessary for the quadratic formula to result in two real roots, one real root and no real solutions.

Ask students to solve each of the following using the quadratic formula.

A: $2x^2 + 10x + 3 = 0$

B: $x^2 + 6x + 9 = 0$

C: $2x^2 + 3x + 5 = 0$

As teachers observe students' work, use the following prompts to promote discussion:

- What is the value under the square root in each equation? What does this tell you about the roots of the equation?
- What connection can be made between the value of $b^2 - 4ac$ and the number of real roots an equation has?
- What values of $b^2 - 4ac$ could lead to approximate answers?

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to solve $x^2 + 2x + 5 = 0$ by completing the square.

(RF5.5)

- When Chantal was asked to describe the roots of the equation $14x^2 - 5x = -5$, she rearranged the equation so that it would equal zero, then used the quadratic formula to find the roots. Her workings are shown below. Edward said that she didn't have to do all that and he then showed the class his work. Ask students if they are both correct. They should identify their preferred method and explain their reasoning.

Chantal	Edward
$x = \dfrac{5 \pm \sqrt{25-280}}{28}$	$b^2 - 4ac$
$x = \dfrac{5 \pm \sqrt{-255}}{28}$	$= 25 - 280$
$x = \dfrac{5 \pm \sqrt{-1}\sqrt{255}}{28}$	$= -255$
$x = \dfrac{5 \pm i\sqrt{255}}{28}$	no real roots
imaginary roots	

(RF5.6)

- Ask students to answer the following:
  - (i) For what values of t does $x^2 + tx + t + 3 = 0$ have one real root?
  - (ii) Show that if the quadratic equation $px^2 + (2p + 1)x + p = 0$ has two real unequal roots, then $4p + 1 > 0$.
  - (iii) Assume a, b and c are real numbers. How many times would $y = ax^2 + bx + c$, intersect the x-axis if the discriminant of $ax^2 + bx + c = 0$ is
    - (a) positive?
    - (b) zero?
    - (c) negative?
  - (iv) Create quadratic equations that have two distinct roots, two equal roots, and imaginary roots. Explain your reasoning.

(RF5.6)

- Ask students to write in simplest form:
  - (i) $\sqrt{-16}$
  - (ii) $\sqrt{-248}$

(RF5.6)

Resources/Notes

Authorized Resource

Pre-Calculus 11

4.4 The Quadratic Formula

SB: pp. 244-256
TR: pp. 160-165
BLM: 4-3, 4-7

Relations and Functions

Outcomes

Students will be expected to

RF5 Continued...

Achievement Indicator:

RF5.7 *Solve a problem by:*

- *analyzing a quadratic equation*
- *determining and analyzing a quadratic equation.*

Elaborations—Strategies for Learning and Teaching

Quadratic equations can be used to model a variety of situations such as projectile motion and geometry based word problems. Students should be exposed to examples which require them to model the problem using a quadratic equation, solve the equation and interpret the solution.

Consider the example:

- A rectangular lawn measuring 8 m by 4 m is surrounded by a flower bed of uniform width. The combined area of the lawn and flower bed is 165 m^2. What is the width of the flower bed?

It is important for students to recognize that the context of the problem dictates inadmissible roots. Discuss with students different scenarios that produce inadmissible roots. For example, time, height and length, would not make sense if they have a negative numerical value. However, temperature could be both negative and positive. When modelling situations for students, emphasize that restrictions sometimes need to be placed on the independent variable of the function. If a solution does not lie in the restricted domain, then it is not a solution to the problem. The following is an example with a restricted domain:

- A baseball is thrown from an initial height of 3 m and reaches a maximum height of 8 m, 2 seconds after it is thrown. At what time does the ball hit the ground?

In the above example, the quadratic equation only models the path of the ball from the time it leaves the throwers' hand to the time it makes first contact with the ground. This quadratic equation yields two possible solutions, one of which is negative. This implies that it occurred before the ball was thrown. The restriction on the domain causes the negative solution to be inadmissible since time cannot be negative and only the positive solution is accepted.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- A ball is thrown from a building at an initial height of 11 metres and reaches a maximum height of 36 metres, 5 seconds after it is thrown. Ask students to do the following:
  (i) Write a quadratic equation which models this situation.
  (ii) Three targets are placed at different locations on the ground. One is at (10,0), another at (11,0) and a final target is placed at (12, 0). Which target does the ball hit? Explain how you arrived at your answer.

  (RF5.7)

- Ask students to find two consecutive whole numbers such that the sum of their squares is 265.

  (RF5.7)

- A diver's path when diving off a platform is given by $d = -5t^2 + 10t + 20$, where d is the distance above the water (in feet) and t is the time from the beginning of the dive (in seconds).
  (i) How high is the diving platform?
  (ii) When is the diver 25 feet above the water?
  (iii) When does the diver enter the water?

  (RF5.7)

- Ask students to choose a quadratic word problem from the class notes or group workstations and use it as a guide to create their own word problem (encourage them to use a real-life situation in which they are interested). Remember to have them include their solutions on a separate sheet. Students could give their problem to another student to solve.

  (RF5.7)

Resources/Notes

Authorized Resource

Pre-Calculus 11

4.4 The Quadratic Formula

SB: pp. 244-257
TR: pp. 160-165
BLM: 4-3, 4-7

Radical Expressions and Equations

Suggested Time: 13 Hours

Unit Overview

Focus and Context

In this unit, students will simplify radical expressions and perform the four operations on these expressions (addition, subtraction, multiplication and division). When working with division, students will be expected to rationalize denominators containing radicals. They will also identify the restrictions on the values of variables in radical expressions that are real numbers.

Students will solve problems that involve radical equations, limited to square roots with non-negative radicands, and identify the extraneous roots.

Outcomes Framework

GCO
Develop algebraic reasoning and number sense.

SCO AN2
Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.

SCO AN3
Solve problems that involve radical equations (limited to square roots with non-negative radicands).

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Algebra and Number	Algebra and Number	Relations and Functions
AN2 Demonstrate an understanding of irrational numbers by: • representing, identifying and simplifying irrational numbers • ordering irrational numbers. [CN, ME, R, V]	AN2 Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands. [CN, ME, PS, R] AN3 Solve problems that involve radical equations (limited to square roots with non-negative radicands). [C, PS, R]	RF12 Graph and analyze radical functions (limited to functions involving one radical). [CN, R, T,V]

Mathematical Processes

[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

Algebra and Number

Outcomes

Students will be expected to

AN2 Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.

[CN, ME, PS, R]

Achievement Indicators:

AN2.1 *Compare and order radical expressions with numerical radicands in a given set.*

AN2.2 *Express an entire radical with a numerical radicand as a mixed radical.*

AN2.3 *Express a mixed radical with a numerical radicand as an entire radical.*

Elaborations—Strategies for Learning and Teaching

In Grade 9, students determined the square root of a perfect square and worked with benchmarks to approximate the square root of non-perfect square rational numbers (9N5, 9N6). In Mathematics 1201, students worked with mixed and entire radicals, limited to numerical radicands (AN2). The notation $\sqrt[n]{x}$ was introduced, where the index of the radical was a maximum of 5.

In this unit, students will simplify radical expressions with numerical and variable radicands and will add, subtract, multiply and divide these expressions. When working with division, students will be responsible for a monomial and binomial denominator. They will also identify values of the variable for which the radical expression is defined.

Students will apply and use radicals to solve equations involving radical expressions. Solutions will be verified by substitution and extraneous roots will be explored.

In Mathematics 1201, students expressed a radical as either a mixed or entire radical with numerical radicands. Review this concept with students and reinforce that if radicals have the same index, the radicands can be compared. It is helpful to rearrange the mixed radical as an entire radical for the purpose of ordering and estimation without the use of technology. Ask students, for example, to determine which is greater, $3\sqrt{5}$ or $4\sqrt{3}$. Although students could use a calculator to approximate the length, the focus here is to rewrite the numerical radicals in equivalent forms and make a comparison.

Part of this unit is a review of topics taught in Mathematics 1201, with the exception of the introduction to secondary roots. Refer to the Roots and Powers unit in the Mathematics 1201 Curriculum Guide for further information regarding teaching strategies and common student errors (AN2.6, AN2.7, AN2.8).

Students should have some exposure to principal and secondary square roots. They should recognize that every positive number has two roots. For example, the square root of 49 is 7 since $7^2 = 49$. Likewise $(-7)^2 = 49$ so -7 is also a square root of 49. The value $\sqrt{49} = 7$ is called the principal square root and $-\sqrt{49} = -7$ is the secondary square root. Although students will compare principal and secondary square roots, it is equally important for them to understand why it makes sense to only use the principal square root in certain situations. For example, if students are using the Pythagorean theorem to calculate the length of a leg of a right triangle, length is a positive number, so the square root must also be positive.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Performance

- Set up a clothesline across the whiteboard to represent a number line with several benchmarks identified. Each student is given a card with an expression of mixed or entire radical. Ask them to pin the card along the number line. They should be able to explain why they placed the card in that position.

  (AN2.1, AN2.2, AN2.3)

- Each group (3-4 students) will be given a deck of ten cards. Each card will have a different mixed radical. The group will then work together to sort the cards from largest to smallest. The first group with the cards sorted in the correct order wins the competition.

  (AN2.1, AN2.2, AN2.3)

- Students can play the *Radical Matching Game* in groups of two. Give students a deck of cards containing pairs that display equivalent mixed radicals and entire radicals. All cards should be placed face down on the table. The first student turns over 2 of the cards, looking for a pair. If they get a pair, they remove the cards and go again. If the overturned cards do not form a pair, it is the other player's turn. The player with the most matches at the end of the game wins.

  (AN2.1, AN2.2, AN2.3)

Resources/Notes

Authorized Resource

Pre-Calculus 11

5.1 Working with Radicals

Student Book (SB): pp. 272-281
Teacher Resource (TR): pp. 179-185

Blackline Master (BLM): 5-3, 5-4

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/rads.html

- cards, with radical expressions, for activities

Algebra and Number

Outcomes

Students will be expected to

AN2 Continued ...

Achievement Indicator:

AN2.4 *Explain, using examples, that* $(-x)^2 = x^2$, $\sqrt{x^2} = |x|$ *and* $\sqrt{x^2} \neq \pm x$; *e.g.,* $\sqrt{9} \neq \pm 3$.

Elaborations—Strategies for Learning and Teaching

When asked to simplify $\sqrt{x^2}$, students may initially conclude that $\sqrt{x^2} = x$. Prompt student discussion using the following questions:

- Does this happen with all positive values of *x*?
- Does this happen with all negative values of *x*?

Ask students to simplify $\sqrt{(-5)^2}$. The following is a sample of student answers:

$\sqrt{(-5)^2} = (-5)^{\frac{2}{2}} = (-5)^1 = -5$	$\sqrt{(-5)^2} = \sqrt{25} = 5$

This is an opportunity for discussion around the correct answer when the value of *x* is negative. Some students may challenge the incorrect solution provided in column one above. Although $\sqrt{(-5)^2}$ may be considered equivalent to $(-5)^{2/2}$, it is important to note the square is completely under the radical sign, so the direction is that the square be evaluated and then the root. Hence, $\sqrt{(-5)^2} = \sqrt{25} = 5$. It is important for students to recognize that when *a* is a negative number, $a^{\frac{m}{n}}$ is not defined because it is not possible to define such expressions consistently. Students can compare this value when *n* is even to when it is odd. (i.e., $(-4)^{\frac{1}{2}} = \sqrt{-4}$ is undefined under the set of real numbers but $(-8)^{\frac{1}{3}} = \sqrt[3]{-8}$ is defined).

This is a great opportunity for discussion as to what is the correct answer when the value of *x* is negative. Although it is common for students to replace $\sqrt{x^2}$ with *x*, students must recognize that this is correct when $x \geq 0$. They should also realize that $\sqrt{x^2}$ is equivalent to $-x$ when $x \leq 0$. Consider the following:

When $x = 4$:

$$\sqrt{x^2} = \sqrt{(4)^2}$$
$$\sqrt{x^2} = \sqrt{16}$$
$$\sqrt{x^2} = 4$$
$$\therefore \sqrt{x^2} = x$$

When $x = -4$:

$$\sqrt{x^2} = \sqrt{(-4)^2}$$
$$\sqrt{x^2} = \sqrt{16}$$
$$\sqrt{x^2} = 4$$
$$\sqrt{x^2} = -x$$
$$\therefore \sqrt{x^2} \neq x$$

Although students will be exposed to the concept of absolute value later in this course, an introduction to absolute value is important here. Ensure students understand the absolute value symbol produces a result that is always positive, which is the principal square root. Hence $\sqrt{x^2} = |x|$. Once this idea has been established, students can assume, for the sake of simplicity, all variables are positive. Therefore $\sqrt{x^2} = x$.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Journal

- Ask students to respond to the following:
  - (i) Your friend stated that since $3^2 = 9$ and $(-3)^2 = 9$, then $\sqrt{x^2}$ is always $\pm x$. Use specific examples to show whether you agree or disagree with his statement.

    (AN2.4)

  - (ii) Explain why $(-16)^{\frac{1}{2}}$ is undefined, where as $(-64)^{\frac{1}{3}}$ is defined under the set of real numbers.

    (AN2.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11

5.1 Working with Radicals

SB: pp. 272-281
TR: pp. 179-185
BLM: 5-3, 5-4

Algebra and Number

Outcomes

Students will be expected to

AN2 Continued ...

Achievement Indicators:

AN2.5 *Identify the values of the variable for which a given radical expression is defined.*

AN2.6 *Express an entire radical with a variable radicand as a mixed radical.*

AN2.7 *Express a mixed radical with a variable radicand as a entire radical.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students simplified radicals with numerical radicands. They also determined the domain of a variety of functions (RF1). This is their first exposure to simplifying expressions with variable radicands. They will write the restrictions on the variable and then write the expression in its simplest form.

Students should be comfortable recognizing square roots of non-negative numbers are defined under the set of real numbers. Remind students of this concept using any number of numerical examples such as $\sqrt{4} = 2$ while $\sqrt{-4}$ is not defined under the set of real numbers. Expose students to examples similar to $\sqrt{16}, \sqrt[4]{-81}, \sqrt[3]{-27}, \sqrt[3]{8}, \sqrt[6]{-64}$. Students should recognize that if a radical has an even index, the radicand must be non-negative. If a radical has an odd index, the radicand can be any real number, including negative numbers.

This could lead into a discussion of what happens if the radicand is variable in nature such as $\sqrt{x}$. Reinforce the concept that the domain of a square root function is limited to values for which the function has meaning. Use examples to allow students to intuitively investigate variable expressions as radicands and then progress to actually solving an inequality algebraically. While students solved inequalities in Grade 9 (9PR4), it may be necessary to review the various rules used to solve inequalities.

Students will convert an entire radical with one variable to a mixed radical and will then progress to multiple variables. They will also work backwards and express a mixed radical with a variable as an entire radical. When simplifying $\sqrt[4]{x^7}$, for example, students can rewrite as $\sqrt[4]{x^4x^3} = x\sqrt[4]{x^3}$. For the radical to represent a real number, $x \geq 0$ because the index is an even number. When writing $2x\sqrt[3]{5x^2}$ as an entire radical, the variable x in the expression $\sqrt[3]{2^3x^35x^2}$ can be any real number since the index of the radical is an odd number.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Observation

- Ask students to complete the following chart and make predictions about the restrictions on each variable.

x	$\sqrt{x}$	$\frac{1}{x}$	$\frac{1}{\sqrt{x}}$		$\sqrt{x-1}$	$\sqrt{x+1}$	$\frac{1}{\sqrt{x-1}}$	$\frac{1}{\sqrt{x+1}}$
-3								
-2								
-1								
0								
1								
2								
3								

As teachers observe students' work, use the following questions to guide discussion:

(i) What values of *x* were undefined? What values of *x* were defined?
(ii) Is the restriction different if the radical expression is in the denominator?
(iii) How could solving inequalities help when determining the restriction?

(AN2.5)

Performance

- For the activity *Sticky Bars*, present students with a selected response question where they could be expected to convert an entire radical with a variable radicand to a mixed radical, or vice versa. The answer is anonymously recorded on a post it note and submitted to the teacher. The teacher or student volunteer arranges the sticky notes on the wall or whiteboard as a bar graph representing the different student responses. Have a discussion regarding why students may have selected the answers they did.

(AN2.6, AN2.7)

Resources/Notes

Authorized Resource

Pre-Calculus 11

5.1 Working with Radicals

SB: pp. 272-281
TR: pp. 179-185
BLM: 5-3, 5-4

Algebra and Number

Outcomes

Students will be expected to

AN2 Continued...

Achievement Indicator:

AN2.8 *Perform one or more operations to simplify radical expressions with numerical or variable radicands.*

Elaborations—Strategies for Learning and Teaching

Students will add and subtract radicals that contain numerical and variable radicands before they move into multiplication and division.

This could be introduced by asking students to add $2x + 3x$. Use leading questions such as the following:

- What process is involved in the addition of monomials?
- How is this expression similar to $2\sqrt{7}+3\sqrt{7}$?
- What are like radicals?
- How can this process be applied to radicals?

The goal is for students to realize adding and subtracting radical expressions is comparable to combining variable expressions with like terms. It is necessary for the radical to have the same index and the same radicand. Students can then apply the same strategies with indices other than 2.

The distributive property can also be applied when simplifying sums and differences of radical expressions. Ask students how to rewrite $2x + 3x$ in another form, namely $(2 + 3)x$. Similarly, $2\sqrt{7}+3\sqrt{7}$ can be expressed as $(2+3)\sqrt{7}=5\sqrt{7}$. Check students' understanding by asking them to express $-4\sqrt{3}$ as the sum of two like radicals.

It is important for students to recognize that, even when adding or subtracting like radicals, the solution can require further simplifying. When subtracting the expression $3\sqrt{8}-7\sqrt{8}$, for example, the solution $-4\sqrt{8}$ can be simplified to $-8\sqrt{2}$. It would be good practice to simplify the radical before like terms are combined. This is especially beneficial when working with large numerical radicands. Students should also be exposed to examples, such as, $2\sqrt{18}+3\sqrt{50}-5\sqrt{2}$ or $4\sqrt[3]{54}+\sqrt[3]{-45}-\sqrt[3]{2}$, where it is necessary to simplify one or more radicals in order to complete the addition and/or subtraction operations.

Common errors occur when adding or subtracting radicals. When asked to add $4+2\sqrt{3}$, for example, they may write $6\sqrt{3}$. Students should be encouraged to check their answers by expressing the sum/difference using the distributive property. Another common error occurs when students incorrectly apply the operations of addition and subtraction to the radicands. The value of $31\sqrt{3}+17\sqrt{3}$, for example, does not equal $48\sqrt{6}$. This error may occur more often once students have been introduced to the multiplication of radicals.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Paper and Pencil

- The voltage V required for a circuit is given by $V = \sqrt{PR}$ where P is the power in watts and R is the resistance in ohms (Ω). How many more volts are needed to light a 100 W bulb than a 75 W bulb if the resistance for both is 100 Ω? They should solve the problem in exact and approximate form.

(AN2.8)

Observation

- Invite students to play WODB (Which One Doesn't Belong?). Show students the following expressions. Each expression could be the one which doesn't belong, but for a different reason. Observe students' reasoning for misconceptions.

$2\sqrt{18}$	$4\sqrt{8}+6\sqrt{50}$
$\sqrt{18}+\sqrt{72}$	$-3\sqrt{18}+4\sqrt{3}$

As students become more familiar with WODB activities, challenge them to complete the following WODB. Observe students' reasoning and engage the class in a discussion about the variety of answers possible.

$\sqrt{50}$	$\sqrt{48}$
?	$\frac{1}{2}\sqrt{108}$

(AN2.8)

Resources/Notes

Authorized Resource

Pre-Calculus 11

5.1 Working with Radicals

SB: pp. 272-281
TR: pp. 179-185
BLM: 5-3, 5-4

Suggested Resource

Resource Link: www.k12pl.nl.ca/curr/10-12/math/2200/links/rads.html

- WODB activities

Algebra and Number

Outcomes

Students will be expected to

AN2 Continued...

Achievement Indicator:

AN2.8 *Continued*

Elaborations—Strategies for Learning and Teaching

Student learning will continue with addition and subtraction involving variable radicands. Ask students the following questions:

- How is adding and subtracting algebraic expressions with radicals similar to adding and subtracting numerical expressions with radical values? Use examples similar to $\sqrt{8x^3} - 4\sqrt{2x}$ where $x \geq 0$, and $-2\sqrt[3]{16x^4} + 5x\sqrt[3]{54x}$ and $2\sqrt{45} - 6\sqrt{20}$ to explain your reasoning.
- What strategy did you use?

Similar to adding and subtracting radicals, students will multiply and divide radicals beginning with numerical radicands. To demonstrate the multiplication property of radicals, reiterate the relationship between a radical and a power with rational exponents.

$$3^{\frac{1}{2}} \times 5^{\frac{1}{2}} = (3 \times 5)^{\frac{1}{2}} = 15^{\frac{1}{2}} \Leftrightarrow \sqrt{3} \times \sqrt{5} = \sqrt{3 \times 5} = \sqrt{15}$$

In Mathematics 1201, students applied the laws of exponents to rational exponents (AN3).

Encourage students to look for a pattern through the use of several examples. The multiplicative property of radicals can then be introduced to students, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ where $a \geq 0$ and $b \geq 0$. This property can also be used to discuss why two radicals with the same index can be multiplied. Although students were exposed to this property in Mathematics 1201, it was used exclusively for the purpose of expressing a radical as a mixed or entire radical (AN2).

Students should be given an opportunity to further explore the product rule for radicals (i.e., $c\sqrt{a} \times d\sqrt{b} = cd\sqrt{ab}$) and the commutative property of multiplication. Ask them to rewrite the expression $2(3^{\frac{1}{2}}) \times 5(6^{\frac{1}{2}})$ using radicals to generalize a pattern. Students' workings may vary but the result should be the same. Consider the following sample:

$$\begin{array}{ll}
2(3^{\frac{1}{2}}) \times 5(6^{\frac{1}{2}}) & 2(\sqrt{3}) \times 5(\sqrt{6}) \\
(2 \times 5)(3^{\frac{1}{2}} \times 6^{\frac{1}{2}}) & (2 \times 5)\left(\sqrt{3} \times \sqrt{6}\right) \\
(10)(3 \times 6)^{\frac{1}{2}} & (10)\left(\sqrt{3 \times 6}\right) \\
10(18)^{\frac{1}{2}} & 10\sqrt{18}
\end{array}$$

Students will also be exposed to examples where the index is not 2. They should be introduced to the rule $c\sqrt[n]{a} \times d\sqrt[n]{b} = cd\sqrt[n]{ab}$, where n is a natural number and c, d, a, and b are real numbers. They should recognize that if n is even then $a \geq 0$ and $b \geq 0$.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Performance

- Create centres in the classroom containing worked solutions of simplifying radical expressions with numerical or variable radicands. Students will participate in a carousel activity where they are asked to move throughout the centres to identify and correct errors. Samples are shown below:

(i) $25\sqrt{5} + 13\sqrt{5}$

$= 38\sqrt{10}$

(ii) $\sqrt{18x^3} + 2\sqrt{8x^3}$

$= 3\sqrt{2x^3} + 4\sqrt{2x^3}$

$= 7\sqrt{4x^6}$

$= 14x^3$

(AN2.8)

Paper and Pencil

- Ask students to simplify each of the following:
  (i) $2\sqrt{18} + 9\sqrt{7} - \sqrt{63}$
  (ii) $6\sqrt{32x^3} - 5\sqrt{8x^3} + 3\sqrt{2x^3}$
  (iii) $-5\sqrt[3]{256x} + \sqrt[3]{192x^4}$

(AN2.8)

Resources/Notes

Authorized Resource
Pre-Calculus 11

5.1 Working with Radicals

SB: pp. 272-281
TR: pp. 179-185
BLM: 5-3, 5-4

5.2 Multiplying and Dividing Radical Expressions

SB: pp. 282-293
TR: pp. 186-193
BLM: 5-3, 5-5

Algebra and Number

Outcomes

Students will be expected to

AN2 Continued...

Achievement Indicator:

AN2.8 *Continued*

Elaborations—Strategies for Learning and Teaching

Advise students that they are less likely to make simplification errors if they simplify radicals before multiplying. An example such as the following could be used to illustrate the two methods:

Multiply first: $\sqrt{80} \times \sqrt{12} = \sqrt{960} = 8\sqrt{15}$

Simplify first: $\sqrt{80} \times \sqrt{12} = 4\sqrt{5} \times 2\sqrt{3} = 8\sqrt{15}$

Ask students which method they prefer and why.

Students will also multiply radicals using the distributive property. It may be helpful to walk them through the similarities between multiplying radical expressions and multiplying polynomials.

In order to avoid common errors, this would be a good opportunity to reinforce the commutative and associative property of multiplication. When multiplying $\sqrt{5} \times 3$, for example, students may write $\sqrt{15}$. They may understand their error if they apply the commutative property to rewrite the expression as $3\sqrt{5}$. Another error occurs when students are asked to multiply an expression such as $3\sqrt{5} \times \sqrt{6}$ and their result is $\sqrt{90}$. This can be avoided if the associative property is used to re-order the expression as $3\left(\sqrt{5} \times \sqrt{6}\right) = 3\sqrt{30}$.

Students will explore how multiplying and dividing algebraic expressions with radicals is similar to multiplying and dividing numerical expressions with radical values. Ask them to use examples similar to $-4\sqrt{12} \times -2\sqrt{18}$ and $(-4\sqrt{x}) \times (-2\sqrt{x^2})$ where $x \geq 0$, and $(-4\sqrt[3]{x}) \times (-2\sqrt[3]{x^2})$ to explain their reasoning.

- What strategy did you use?
- When is it necessary to use the distributive property to multiply expressions that contain radicals? Create an example and show the solution.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to simplify the following:

  (i) $(\sqrt{2}+\sqrt{6})^2$

  (ii) $(3\sqrt{8}-4)(2+7\sqrt{3})$

  (iii) $(\sqrt{20}+\sqrt{24})(3\sqrt{12}-5\sqrt{32})$

  (iv) $-2\sqrt[3]{12}(4\sqrt[3]{2}-5\sqrt[3]{9})$

  (v) $(-3\sqrt{x})(6\sqrt{x^3})$

  (vi) $(3\sqrt{x}+2)(3-5\sqrt{x})$

(AN2.8)

Performance

- Ask students to participate in *Commit and Toss.* Provide students with a selected response problem as shown below. They anonymously commit to an answer and provide a justification for the answer they selected. Students crumble their solutions into a ball and toss the papers into a basket. Once all papers are in the basket, ask students to reach in and take one out. Ask students to then move to the corner of the room designated to match their selected response. In their respective corners, they should discuss the similarities or differences in the explanations provided and report back to the class.

> Express $(\sqrt{3}-\sqrt{2})^2$ in simplest form.
>
> (A) 1
> (B) $5-2\sqrt{6}$
> (C) $1-2\sqrt{3}$
> (D) 5
>
> Explain your reasoning:

(AN2.8)

Resources/Notes

Authorized Resource

Pre-Calculus 11

5.2 Multiplying and Dividing Radical Expressions

SB: pp. 282-293
TR: pp. 186-193
BLM: 5-3, 5-5

Suggested Resource

Resource Link: www.k12pl.nl.ca/curr/10-12/math/2200/links/rads.html

- classroom clip demonstrating students performing operations to simplify a radical expression using the activity *Commit and Toss*

Algebra and Number

Outcomes

Students will be expected to

AN2 Continued...

Achievement Indicator:

AN2.9 *Rationalize the denominator of a rational expression with monomial or binomial denominators.*

Elaborations—Strategies for Learning and Teaching

Division of radicals and the rationalization of the denominator are new concepts for students. The denominator of the rational expression can be a monomial or a binomial.

The rules of exponents should be integrated when introducing the division of radicals. For example:

$$\left(\frac{4}{9}\right)^{\frac{1}{2}} = \frac{4^{\frac{1}{2}}}{9^{\frac{1}{2}}} \Leftrightarrow \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} \qquad \left(\frac{8}{27}\right)^{\frac{1}{3}} = \frac{8^{\frac{1}{3}}}{27^{\frac{1}{3}}} \Leftrightarrow \sqrt[3]{\frac{8}{27}} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}}$$

Ask students to predict the rule when dividing radicals. Using the examples, reinforce to students that they can only divide radicals that have the same index. They should recognize the quotient rule of radicals states that the n^{th} root of a quotient is the quotient of the n^{th} root. In other words, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$ where n is a natural number, a and b are real numbers, and $b \neq 0$. Remind students if n is even then $a \geq 0$ and $b > 0$.

Students should be exposed to a variety of cases when simplifying radicals with fractions. Include examples such as the following:

$$\sqrt{\frac{25}{4}} = \frac{\sqrt{25}}{\sqrt{4}} = \frac{5}{2} \qquad \frac{\sqrt{12}}{\sqrt{6}} = \sqrt{\frac{12}{6}} = \sqrt{2} \qquad \sqrt{\frac{12}{5}} = \frac{\sqrt{12}}{\sqrt{5}} = \frac{2\sqrt{3}}{\sqrt{5}}$$

As students simplify radicals such as these, they should ask themselves the following questions:

- Is the denominator a perfect root?
- Can the numerator and denominator divide into a rational number?
- Will the denominator have a radical when simplified?

Students will develop a strategy for converting a fraction that has radicals in its denominator into an equivalent fraction with no radicals in the denominator. Rationalizing the denominator provides a standard notation for expressing results. Using an example such as $\frac{2\sqrt{3}}{\sqrt{5}}$, ask them what they can multiply the numerator and denominator by that results in a rational expression.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Performance

- In groups of two, ask students to participate in the activity *Pass the Problem.* Give each pair a simplification problem that involves rationalizing the denominator. Ask one student to write the first line of the solution and pass it to the second student. The second student will verify the workings and check for errors. If there is an error present, ask students to discuss the error and why it occurred. The student will then write the second line of the solution and pass it their partner. This process continues until the solution is complete.

(AN2.9)

Resources/Notes

Authorized Resource

Pre-Calculus 11

5.2 Multiplying and Dividing Radical Expressions

SB: pp. 282-293
TR: pp. 186-193
BLM: 5-3, 5-5

Algebra and Number

Outcomes

Students will be expected to

AN2 Continued...

Achievement Indicators:

AN2.9 *Continued*

AN2.10 *Describe the relationship between rationalizing a binomial denominator of a rational expression and the product of the factors of a difference of squares expression.*

Elaborations—Strategies for Learning and Teaching

Expose students to examples where the index is greater than 2. For example, when students simplify $\frac{\sqrt[3]{2}}{\sqrt[3]{5}}$, they may initially multiply the numerator and denominator by the term $\sqrt[3]{5}$. They should recognize a radical still remains in the denominator. When rationalizing the denominator, the root should guide student choices. They multiply numerator and denominator by a value where the powers of the denominator will equal the index of the root. Therefore, $\frac{\sqrt[3]{2}}{\sqrt[3]{5^1}}\left(\frac{(\sqrt[3]{5^2}}{(\sqrt[3]{5^2}}\right)$ results in a rational denominator.

This strategy can be applied to variable monomial denominators. Given an example, such as $\frac{\sqrt[4]{7x}}{\sqrt[4]{27x^2y^2}}$, the index will help students determine what expression they should use to rationalize the denominator. In other words, $\frac{\sqrt[4]{7x}}{\sqrt[4]{3^3x^2y^2}}\left(\frac{\sqrt[4]{3^1x^2y^2}}{\sqrt[4]{3^1x^2y^2}}\right)$ will produce a rational denominator.

Examples such as $\frac{2\sqrt{3}}{7\sqrt{5}}$, where the denominator is a mixed radical should be included. Encourage students to think about what they would have to multiply by to rationalize the denominator. The initial tendency may be to multiply the numerator and denominator by $7\sqrt{5}$. Although it is not the most efficient strategy, it is correct. However, the resulting expression will have to be simplified.

They should realize multiplying by $\frac{\sqrt{5}}{\sqrt{5}}$ will rationalize the denominator. Students should first be exposed to expressions with entire radicals in both the numerator and denominator. This should then be extended to include mixed radicals and examples where there is more than one term in the numerator. It is the students' choice whether they simplify before or after they rationalize the denominator. When working with larger numbers, however, simplifying first would allow them to work with smaller numbers.

Students will simplify an expression such as $\frac{1}{2+\sqrt{3}}$ where it is necessary to rationalize a binomial denominator. Initially, they may think they can multiply the numerator and denominator by $\sqrt{3}$, as in their previous work with monomial denominators. As they explore this, however, they should discover that $\frac{1}{2+\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{2\sqrt{3}+3}$, and the denominator still contains a radical. Prompt them to multiply the numerator and denominator by the conjugate of $2+\sqrt{3}$. This is similar to multiplying the factors of a difference of squares expression. Students should also be exposed to expressions where both terms in the binomial are irrational. Expressions involving variable numerators and denominators should be explored in a similar fashion.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to rationalize the denominator for each of the expressions:

  (i) $\dfrac{\sqrt{7}}{1+\sqrt{7}}$

  (ii) $\dfrac{2+3\sqrt{5}}{2\sqrt{5}-4}$

  (iii) $\dfrac{b}{a+\sqrt{b}}$

  (iv) $\dfrac{x+3\sqrt{y}}{\sqrt{y}-x}$

(AN2.9, AN2.10)

Journal

- When asked to rationalize the denominator in the expression $\frac{4}{2+\sqrt{7}}$ your friend said he could just multiply the expression by $\frac{\sqrt{7}}{\sqrt{7}}$. Ask students to explain why this would not work.

(AN2.9, AN2.10)

Resources/Notes

Authorized Resource

Pre-Calculus 11

5.2 Multiplying and Dividing Radical Expressions

SB: pp. 282-293
TR: pp. 186-193
BLM: 5-3, 5-5

Algebra and Number

Outcomes

Students will be expected to

AN3 Solve problems that involve radical equations (limited to square roots with non-negative radicands).

[C, PS, R]

Achievement Indicators:

AN3.1 *Determine any restrictions on values for the variable in a radical equation.*

AN3.2 *Determine the roots of a radical equation algebraically, and explain the process used to solve the equation.*

AN3.3 *Verify, by substitution, that the values determined in solving a radical equation algebraically are roots of the equation.*

AN3.4 *Explain why some roots determined in solving a radical equation algebraically are extraneous.*

Elaborations—Strategies for Learning and Teaching

Students will solve radical equations involving square roots. It is intended that the equations will have no more than two radicals and the radicand will contain variables that are first and second degree. Students will be responsible for solving equations resulting in a linear or quadratic equation.

Earlier in this unit, students determined restrictions within the real number system on a variable in a radical that had an even index. They will continue to determine the restrictions before solving a radical equation.

This would be a good opportunity for students to compare a radical equation to its graph to develop an understanding of restrictions for the variable and the points that satisfy the equation. Using a table of values, ask students to graph $y = \sqrt{x}$.

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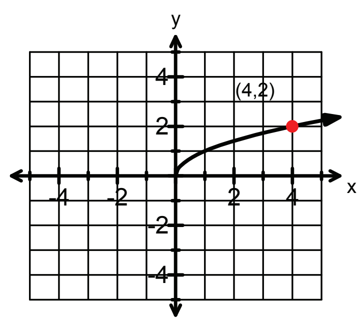
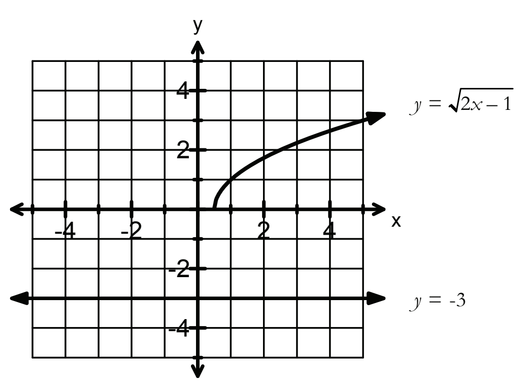


Focusing on the point (4,2), ask students how they would algebraically solve the equation given only the *y*-coordinate 2. When solving $2 = \sqrt{x}$, the value of *x* can be determined by inspection. Students could also use the idea that squaring a number is the inverse operation of taking the square root. This technique may seem straight forward but students should be exposed to equations where the value is not a solution to the original equation. Consider the example $\sqrt{2x-1} = -3$. The left-hand side of the equation calls for a positive square root, but the right-hand side of the equation is negative. Intuitively, there can be no solution.



General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to explain why the domain of $\sqrt{2x-5}$ is $x \geq \frac{5}{2}$, while the domain of $\frac{1}{\sqrt{2x-5}}$ is $x > \frac{5}{2}$. They should illustrate their answers graphically with technology and then algebraically.

(AN3.1)

- Ask students to state the restrictions, solve and check for extraneous roots.
  (i) $\sqrt{4x} = 8$
  (ii) $\sqrt{x+4} = 5$
  (iii) $\sqrt{2x-3} = -2$

(AN3.1, AN3.2, AN3.3)

Resources/Notes

Authorized Resource

Pre-Calculus 11

5.3 Radical Equations

SB: pp. 294-303
TR: pp. 194-199
BLM: 5-3, 5-6

Algebra and Number

Outcomes

Students will be expected to

AN3 Continued...

Achievement Indicators:

AN3.1, AN3.2

AN3.3, AN3.4 *Continued*

Elaborations—Strategies for Learning and Teaching

Students should recognize that given the *y*-coordinate of -3, there is no possible *x*-coordinate that satisfies the equation. However, squaring both sides of the equation results in $x = 5$. This cannot be correct, as both substitution and the graph have shown that the equation has no solution. This is a great lead in to the concept of extraneous roots.

Extraneous roots occur because squaring both sides and solving the equation may result in roots that do not satisfy the original equation. As students solve equations, reinforce the importance of checking that the value is a solution to the original equation. Any extraneous roots are rejected as answers.

Another strategy students can use to solve a radical equation involves applying a power that will eliminate the radical expression on both sides of the equation. To solve $\sqrt{x+1} = 4$, for example, students would first rewrite the radical expression with a rational exponent, resulting in $(x+1)^{\frac{1}{2}} = 4$. Using the multiplicative inverse gives $\left((x+1)^{\frac{1}{2}}\right)^2 = (4)^2$ and leads to an equation without radicals.

Students should be exposed to equations where the radical is not isolated. In such situations, make a comparison to solving a linear equation. They should recognize that solving an equation such as $3+\sqrt{2x+1} = 7$ follows a process that is similar to solving $3 + x = 7$.

Rather than squaring both sides of a radical equation, students sometimes mistakenly square the individual terms. When solving $3+\sqrt{2x+1} = 7$, for example, they may not isolate the radical. Squaring each term results in the incorrect equation $3^2 + (\sqrt{2x+1})^2 = 7^2$.

The following illustration could be used to reinforce why squaring individual terms of an equation is not the same as squaring both sides of the equation.

$3 + 4 = 7$ $3^2 + 4^2 = 7^2$ $9 + 16 = 49$ $25 \neq 49$	$3 + 4 = 7$ $(3 + 4)^2 = 7^2$ $7^2 = 7^2$ $49 = 49$

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Performance

- Ask students to participate in *Commit and Toss.* Provide students with a selected response problem as shown below. They anonymously commit to an answer and provide a justification for the answer they selected. Students crumble their solutions into a ball and toss the papers into a basket. Once all papers are in the basket, ask students to reach in and take one out. Ask students to then move to the corner of the room designated to match their selected response. In their respective corners, they should discuss the similarities or differences in the explanations provided and report back to the class.

> What is the value of x in the equation $\sqrt{-3x+6}=6$?
>
> (A) -10
> (B) -2
> (C) 0
> (D) 10
>
> Explain your reasoning:

(AN3.1, AN3.2, AN3.3)

Resources/Notes

Authorized Resource

Pre-Calculus 11

5.3 Radical Equations

SB: pp. 294-303
TR: pp. 194-199
BLM: 5-3, 5-6

Algebra and Number

Outcomes

Students will be expected to

AN3 Continued...

Achievement Indicators:

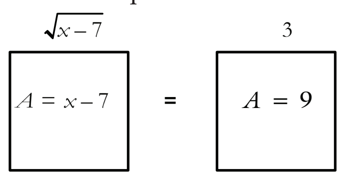
AN3.1, AN3.2

AN3.3, AN3.4 *Continued*

Elaborations—Strategies for Learning and Teaching

The area model can also be used to solve radical equations. In Grade 8, students viewed the area of the square as the perfect square number, and the side length of the square as the square root (8N1). Recall that if a square has an area of 4, then its side has a length of 2. Similarly, if a square has an area of 3, then its side has a length of $\sqrt{3}$. Teachers should prompt discussion about the side length of a square if its area is x. Consider the following example: Solve $\sqrt{x-7}=3$.

Using the area model, students label the dimensions of one square as $\sqrt{x-7}$ and the dimension of the other square as 3. They can then determine the area of each square.

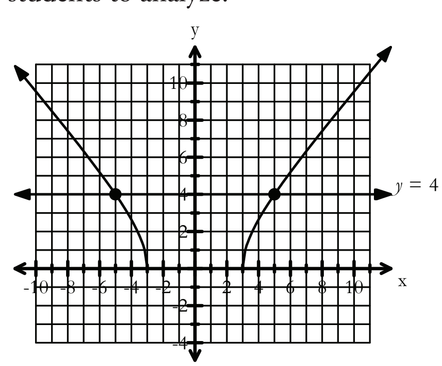


Students should recognize that the goal is to determine the value of x which results in the same area for both squares. Solving the equation $x - 7 = 9$ results in $x = 16$. This representation helps students visualize what each equation is describing. Encourage them to check their answers by substituting the value back into the original equation.

Students will also solve radical equations that result in quadratic equations. Use an example such as $\sqrt{x^2-9}=4$ and the following questions to promote student discussion:

- How is the radicand similar yet different to those studied to date?
- How many solutions might this equation have?

Using technology, present the graph of the corresponding function $y=\sqrt{x^2-9}$ for students to analyze.



From the graph, students should observe that the domain for the function is $x \geq 3$ or $x \leq -3$. This will help them determine whether $x = \pm 5$ are the solutions to the equation.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to respond to the following:

  The following steps show how a student solved the equation $3+2\sqrt{n+4}=5$. Ask students if the final answer is correct and whether the student should receive full marks for the solution. They should justify their answer.

  $3+2\sqrt{n+4}=5$

  $5\sqrt{n+4}=5$

  $\sqrt{n+4}=1$

  $\left(\sqrt{n+4}\right)^2=1^2$

  $n+4=1$

  $n=-3$

  (AN3.1, AN3.2, AN3.3)

- Ask students to solve $6=\sqrt{2n+15}-n$ and check for extraneous roots.

  (AN3.1, AN3.2, AN3.3, AN3.4)

Resources/Notes

Authorized Resource
Pre-Calculus 11

5.3 Radical Equations

SB: pp. 294-303
TR: pp. 194-199
BLM: 5-3, 5-6

Algebra and Number

Outcomes

Students will be expected to

AN3 Continued...

Achievement Indicators:

AN3.1, AN3.2

AN3.3, AN3.4 *Continued*

AN3.5 *Solve problems by modeling a situation using a radical equation.*

Elaborations—Strategies for Learning and Teaching

Students should proceed to solve the radical equation algebraically by squaring both sides of the equation. The only difference is the resulting equation is quadratic. They will be able to solve the quadratic using the method of their choice (i.e., graphing, factoring, quadratic formula).

Encourage students to continue to substitute the value of x back into the original equation to determine if there are extraneous roots. When students solve $\sqrt{4x+17} = x+3$, for example, the solutions are x = -4 and x = 2. The value of x = -4 results in 1= -1, which is a false statement. Therefore, there is only one solution x = 2.

Students will also solve equations that involve two radical expressions. As students square both sides in an equation such as $\sqrt{x+7} = \sqrt{x} + 1$ they should recognize the resulting equation still contains a radical. Therefore, they will need to repeat the process of isolating the radical term and squaring both sides of the equation again. Continue to remind students of the importance of checking for extraneous roots when solving radical equations.

Students will be exposed to application problems where the equation may contain a radical that is a square root. They will solve for the unknown variable by squaring both sides of the equation. Provide students with the following example and ask them to answer the questions:

- Collision investigators can approximate the initial velocity, v, in kilometres per hour, of a car based on the length, l, in metres, of the skid mark. The formula $v = 12.6\sqrt{l} + 8$, where $l \geq 0$ models the relationship. What length of skid is expected if a car is travelling 50 km/hr when the brakes are applied? How is knowledge of radical equations used to solve this problem?

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to solve each of the following equations. Remind them to check for extraneous roots.

  (i) $\sqrt{3x-5}-2=3$

  (ii) $\sqrt{2x+4x}=3-\sqrt{2x}$

  (iii) $\sqrt{x+7}=\sqrt{x}+1$

  (iv) $\sqrt{x^2-8x}=3$

(AN3.1, AN3.2, AN3.3)

- Ask students to answer the following:

  (i) The period T (in seconds) is the time it takes a pendulum to make one complete swing back and forth. This is modelled by $T=2\pi\sqrt{\frac{L}{32}}$, where L is the length of the pendulum in feet. Ask students to determine the period of the pendulum if its length is 2 ft.

  (ii) The radius of a cylinder can be found using the equation $r=\sqrt{\frac{V}{\pi \text{h}}}$ where r is the radius, V is the volume, and h is the height. A cylindrical tank can hold 105.62 m^3 of water. If the height of the tank is 2 m, what is the radius of its base?

  (iii) The surface area (S) of a sphere with radius r can be found using the equation $S = 4\pi r^2$.

  (a) Using the given equation, how could you find the radius of a sphere given its surface area? Write the equation.

  (b) The surface area of a ball is 426.2 cm^2. What is its radius?

(AN3.5)

Resources/Notes

Authorized Resource

Pre-Calculus 11

5.3 Radical Equations

SB: pp. 294-303
TR: pp. 194-199
BLM: 5-3, 5-6

Rational Expressions and Equations

Suggested Time: 13 Hours

Unit Overview

Focus and Context

In this unit, students will simplify a rational expression and determine the non-permissible values. They will perform the operations on rational expressions (addition, subtraction, multiplication, division).

Students will solve problems that involve rational equations. They will determine the solution to a rational equation algebraically and identify the non-permissible values.

Outcomes Framework

GCO

Develop algebraic reasoning and number sense.

SCO AN4

Determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).

SCO AN5

Perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).

SCO AN6

Solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials or trinomials).

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Algebra and Number	Algebra and Number	Relations and Functions
AN5 Demonstrate an understanding of common factors and trinomial factoring,concretely, pictorially and symbolically. [C, CN, R, V]	AN4 Determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials). [C, ME, R] AN5 Perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials). [CN, ME, R] AN6 Solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials or trinomials). [C, PS, R]	RF12 Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials). [CN, R, T, V]

Mathematical Processes

[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

Algebra and Number

Outcomes

Students will be expected to

AN4 Determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).

[C, ME, R]

Achievement Indicators:

AN4.1 *Explain why a given value is non-permissible for a given rational expression.*

AN4.2 *Determine the non-permissible values for a rational expression.*

Elaborations—Strategies for Learning and Teaching

In Grade 9, students solved problems that involved arithmetic operations on rational numbers (9N3). They will now be introduced to rational expressions limited to numerators and denominators that are monomials, binomials and trinomials. They will simplify them and determine the non-permissible values.

A rational expression is any expression that can be written as the quotient of two polynomials, in the form $\frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$. To begin work with rational expressions, provide students with several examples of expressions, such as $\frac{4}{5}, \frac{2x}{y}, \frac{x^2-4}{x+1}, \sqrt{5}, 2\pi, \frac{\sqrt{x}}{2y}$ and ask them to identify and explain why an expression is or is not a rational expression. It should be pointed out to students that all rational expressions are algebraic fractions but not all fractions are rational expressions. In the above list, for example, $\frac{4}{5}, \sqrt{5}$, and 2π are not rational expressions.

Non-permissible values are the values of a variable that make the denominator of a rational expression equal zero. In Grade 7, students were introduced to the concept of why a number cannot be divided by zero (7N1). Students should first find the non-permissible values of a rational expression where the denominator is a first degree polynomial and then progress to second degree polynomials. Given the expression, $\frac{x}{x+2}$, for example, use the following to promote student discussion around non-permissible values:

- Using inspection, what value of x would make the denominator zero?
- Explain why this value of x is called a non-permissible value.
- Fill in the following table of values. What do you notice? What is the domain of the expression $\frac{x}{x+2}$?

x	-3	-2	-1	0	1	2	3
y							

- Write a general rule for determining the non-permissible value for a first degree denominator.
- Can the numerator be equal to zero? Explain your reasoning.

Students should notice that the non-permissible value of $\frac{x}{x+2}$ is -2. This can be written as $\frac{x}{x+2}$, $x \neq -2$.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to write a rational expression for the following non-permissible values of 0, -2 and 3. They should compare their answers with the class.

(AN4.2)

Interview

- Ask students to explain why $x = 2$ is a non-permissible value for $\frac{3x}{x-2}$.

(AN4.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11

6.1 Rational Expressions

Student Book (SB): pp. 310-321
Teacher Resource (TR): pp. 209-216

Blackline Master (BLM): 6-4, 6-5

Algebra and Number

Outcomes

Students will be expected to

AN4 Continued ...

Achievement Indicators:

AN4.1, AN4.2 *Continued*

Elaborations—Strategies for Learning and Teaching

A common error occurs when students generalize that the non-permissible value is zero rather than looking at the value(s) of x that produce a denominator of zero. Encourage them to substitute the non-permissible value(s) for x back into the denominator to verify the denominator results in zero.

Students should also be exposed to rational expressions where the denominator is a second degree polynomial. Discuss an expression, such as $\frac{x-1}{3x^2-12}$. Ask students to answer the following questions:

- Using inspection, what value(s) of x would make the denominator zero?
- What other strategies can be used to solve the quadratic equation?
- Are there any rational expressions without non-permissible values?

To solve $3x^2-12 = 0$, students may remove the greatest common factor (GCF) and apply the zero product property, apply the quadratic formula or the square root property. Question students as to which method is more efficient and why.

If students use the square root property, they divide the equation $3x^2 = 12$ by 3 and solve $x^2 = 4$. Some may mistakenly write the non-permissible value as $x = 2$ rather than $x = \pm 2$. Another error occurs when students factor $3x^2 - 12 = 0$ as $3(x - 2)(x + 2)$ and include 3 as a non-permissible value. They may also incorrectly factor the expression in the denominator, whether it is a binomial or trinomial. Remind them to verify their work by expanding the product of the factors using the distributive property.

Students should be exposed to rational expressions with more than one variable. If the expression $2x - 3y$ is in the denominator, for example, the non-permissible values can be found by solving the equation $2x - 3y = 0$ for x or y. Students have this option unless it is specifically stated in the problem to solve for a particular variable.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Performance

- Ask students to work in groups to participate in the *Domino Game*. Provide each group with 10 domino cards. One side of the card should contain a rational expression, while the other side contains non-permissible values for a different rational expression. The task is for students to lay the dominos out such that the non-permissible values on one card will match with the correct rational expression on another. They will eventually form a complete loop with the first card matching with the last card. A sample is shown below:

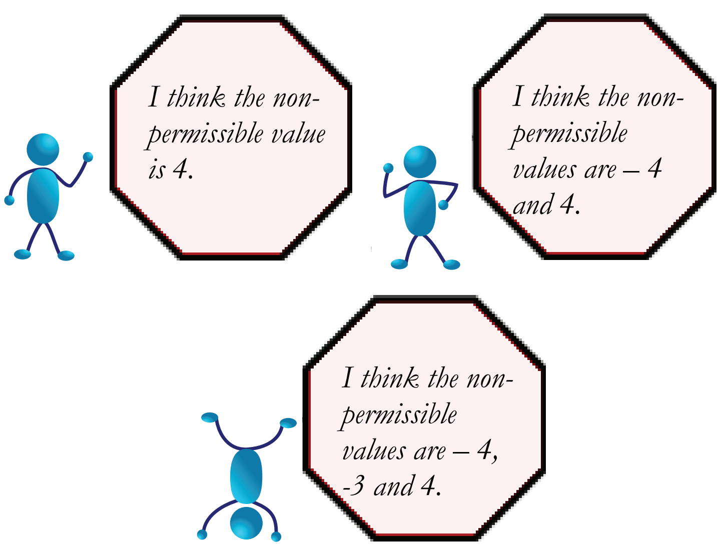
$\frac{3x}{2x-1}$	$x \neq \frac{1}{3}$	$\frac{x+2}{3x-1}$	$x \neq -\frac{1}{2}$	$\frac{7x}{2x+1}$	$x \neq 2$

(AN4.1, AN4.2)

Journal

- Ask students to respond to the following:

What are the non-permissible values for $\frac{x+3}{x^2-16}$?



Who is correct? Justify your answer by solving the problem.

(AN4.1, AN4.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11

6.1 Rational Expressions

SB: pp. 310-321
TR: pp. 209-216
BLM: 6-4, 6-5

Algebra and Number

Outcomes

Students will be expected to

AN4 Continued ...

Achievement Indicators:

AN4.1, AN4.2 *Continued*

AN4.3 *Compare the strategies for writing equivalent forms of rational expressions to the strategies for writing equivalent forms of rational numbers.*

AN4.4 *Determine a rational expression that is equivalent to a given rational expression by multiplying the numerator and denominator by the same factor (limited to a monomial or a binomial), and state the non-permissible values of the equivalent rational expression.*

Elaborations—Strategies for Learning and Teaching

It is important for students to differentiate between non-permissible values and inadmissible values. Remind students that inadmissible values were discussed in the Quadratics Unit when they used quadratic functions to model situations. These are values that do not make sense in a given context. Students will continue to work with inadmissible values for a variable in a rational expression. If a boat traveled 20 km with a speed of x km/h, for example, the time taken for the trip would be represented by $\frac{20}{x}$. If students are asked to determine the slowest speed the boat can travel, they should recognize that the non-permissible value is 0 but that all $x < 0$ are inadmissible.

In Grade 7, students developed skills in writing equivalent positive rational numbers (7N7). Students will apply these strategies to rational expressions. This concept is essential when adding and subtracting rational expressions later in this unit.

Students should recognize that they can multiply or divide a rational expression by 1 without changing its value. A rational expression is not equivalent to another rational expression if their restrictions are different. Guide students through the following activity:

- Consider the rational number $\frac{3}{2}$. Ask students to write a rational number by multiplying both the numerator and denominator by 3, by 5, and by -4? Did the value of their fraction change?
- Consider the rational expression $\frac{4}{x}$ where $x \neq 0$. Ask students to write a rational expression by multiplying both the numerator and denominator by 2, by x, and by $x + 1$. Did any of their expressions produce a new restriction?

Using substitution, ask students to verify if the expressions are equivalent. When the expressions $\frac{4}{x}$, $x \neq 0$ and $\frac{4(x+1)}{x(x+1)}$, $x \neq 0,-1$ are compared, they are both undefined at $x = 0$. When $x = -1$, however, the expression $\frac{4}{x}$ simplifies to -4 while the expression $\frac{4(x+1)}{x(x+1)}$ is undefined. Since the expressions are not equal for the same value of x, the expressions are not equivalent.

Although graphing rational functions is not an outcome in this course, teachers could, as an alternative, prove rational expressions are equivalent for all permissible values of the variable by showing students the graphs of these functions.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to complete the following table:

Are the expressions equivalent?	Yes	No	Justify your choice
$\dfrac{x+3}{x-4}$ and $\dfrac{4x+12}{4x-16}$			
$\dfrac{5}{x-5}$ and $\dfrac{5x+25}{x^2-25}$			
$\dfrac{x+2}{x-3}$ and $\dfrac{3x+6}{2x-6}$			

(AN4.3, AN4.4)

- Ask students which expression is equivalent to $\dfrac{x-3}{x+2}$.

  (A) $\dfrac{x^2-3x}{x^2+2x}$

  (B) $\dfrac{6x-18}{x+2}$

  (C) $\dfrac{3x-3}{3x+2}$

  (D) $\dfrac{4x-12}{4x+8}$

(AN4.3, AN4.4)

Journal

- Your friend thinks the expressions $\frac{x-3}{2x}$ and $\frac{(x-3)(x+1)}{2x(x+1)}$ are equivalent. Ask students to explain why these expressions are not equivalent.

(AN4.3, AN4.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11

6.1 Rational Expressions

SB: pp. 310-321
TR: pp. 209-216
BLM: 6-4, 6-5

Note:
The resource *Pre-Calculus 11* discusses equivalent rational expressions on p. 313. It is important to note that $\frac{7s}{s-2}$ and $\frac{7s^2}{s(s-2)}$ are not equivalent rational expressions since a new restriction has been added.

Algebra and Number

Outcomes

Students will be expected to

AN4 Continued ...

Achievement Indicators:

AN4.3, AN4.4 *Continued*

AN4.5 *Simplify a rational expression.*

AN4.6 *Explain why the non-permissible values of a given rational expression and its simplified form are the same.*

Elaborations—Strategies for Learning and Teaching

When writing an equivalent expression, caution students to use the distributive property appropriately. When simplifying $\frac{x}{x+4}\times\left(\frac{2}{2}\right)$, for example, students may incorrectly write $\frac{2x}{2x+4}$ or $\frac{2x}{x+8}$. To avoid this error, encourage students to place brackets around the binomial when multiplying.

Simplifying a rational expression to lowest terms mirrors the process of simplifying fractions. In both cases, common factors in the numerator and denominator form a ratio of one and can be simplified.

Discuss with students the benefit of simplifying rational expressions, whether it be for evaluating or performing operations. Ask students to evaluate the expression $\frac{x^2+4x}{x}$ where $x \neq 0$ at $x = 2$. Then evaluate the expression $x + 4$ where $x \neq 0$ at $x = 2$. Students should answer the following questions:

- What is the result when substituting the value into the original expression?
- What is the result when substituting the value into the simplified expression?
- Why were the results the same?
- What is the benefit of simplifying an expression before substituting values for the variables?
- Why does the simplified expression include a non-permissible value?

Students should recognize one of the benefits of simplifying an expression is to create an equivalent expression that is easier to evaluate. Ask them why the domain of a rational expression is always determined prior to eliminating duplicate factors in the numerator and denominator. A simplified rational expression, for example, may not have any non-permissible values. However, the simplified expression must retain the non-permissible values of the original expression for both to be equivalent.

Students should be exposed to techniques which make the simplification process more efficient. For example, $\frac{a-b}{b-a}=\frac{-b+a}{b-a}=\frac{-(b-a)}{(b-a)}=-1$.

Once this has been established, students should recognize, for example, that $\frac{x-1}{1-x}=-1$, and continue efficiently from this point.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Performance

- Ask students to participate in the following activities:
  - (i) Divide the class into two groups. One group will be given rational expressions and the other group will be given the associated rational expression in simplest form. Ask students to find a partner who has a rational expression equivalent to theirs.
  - (ii) Ask students to create a unique 3 × 3 Bingo card for Rational Expression Bingo. Distribute a blank Bingo card to each student. Teachers should predetermine various expressions involving rational expressions they would like students to simplify. The expressions should be placed in a bag, with simplified expressions on the board. Ask students to write one of the simplified expressions in each square. The center square should remain a "free" space. The teacher pulls an expression from a bag. Students simplify the expression, find its value on their card and cross it off. The first person with a straight line or four corners wins, or the first person with an X or a T on the Bingo card could win.

(AN4.5)

Paper and Pencil

- Ask students to simplify $\frac{8-2x^2}{2x-4}$ and state the non-permissible values.

(AN4.5)

- Invite students to play WODB (Which One Doesn't Belong?). Show students the following expressions. Each expression could be the one which doesn't belong, but for a different reason. Observe students' reasoning for misconceptions.

$\frac{5}{3-x}$	$\frac{x}{x-1}$
$\frac{x}{\sqrt{x-3}}$	$\frac{x+3}{x^2-9}$

(AN4.2, AN4.5)

Resources/Notes

Authorized Resource

Pre-Calculus 11

6.1 Rational Expressions

SB: pp. 310-321
TR: pp. 209-216
BLM: 6-4, 6-5

Suggested Resource

Resource Link: www.k12pl.nl.ca/curr/10-12/math/2200/links/rats.html

- 3 × 3 Bingo card
- printable file of rational expressions
- WODB expressions

Algebra and Number

Outcomes

Students will be expected to

AN4 Continued...

Achievement Indicator:

AN4.7 *Identify and correct errors in a given simplification of a rational expression, and explain the reasoning.*

Elaborations—Strategies for Learning and Teaching

It is beneficial to have students analyze solutions that contain errors. Along with providing the correct solutions, they should be able to identify incorrect solutions, including why errors might have occurred and how they can be corrected.

When simplifying rational expressions, students may cancel terms rather than factors. They may simplify, for example, $\frac{x^2+x}{x^2-1}$ as $\frac{\cancel{x^2}+x}{\cancel{x^2}-1}$ resulting in *-x*. To help students see this error ask them to make a comparison with a numerical rational expression such as $\frac{8}{12}=\frac{5+3}{5+7}$ and $\frac{8}{12}=\frac{\cancel{5}+3}{\cancel{5}+7}=\frac{3}{7}$. Ask them if $\frac{8}{12}$ is equal to $\frac{3}{7}$. Students should realize that cancelling a portion of the factor is incorrect. Another error occurs when students omit a numerator of 1 after the rational expression is simplified. They mistakenly simplify $\frac{3}{6x}$, for example, as $2x$. Encourage students to check the reasonableness of their answer by rewriting the the expression as $\frac{3}{6}\bullet\frac{1}{x}$.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Observation

- Set up centres containing examples of incorrect simplified rational expressions and their non-permissible values. Ask students to move around the centres to identify and correct the errors. A sample is shown below:

$$\frac{8x-12}{6x^2-4x}, x \neq 0, \tfrac{2}{3}$$

$$\frac{4(2x-3)}{2x(3x-2)}$$

$$\frac{4}{2x}(1)$$

$$2x, x \neq 0, \tfrac{2}{3}$$

(AN4.7)

Resources/Notes

Authorized Resource

Pre-Calculus 11

6.1 Rational Expressions

SB: pp. 310-321
TR: pp. 209-216
BLM: 6-4, 6-5

Algebra and Number

Outcomes

Students will be expected to

AN5 Perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials.

[CN, ME, R]

Achievement Indicators:

AN5.1 *Compare the strategies for performing a given operation on rational expressions to the strategies for performing the same operation on rational numbers.*

AN5.2 *Determine the non-permissible values when performing operations on rational expressions.*

AN5.3 *Determine, in simplified form, the product or quotient of rational expressions.*

Elaborations—Strategies for Learning and Teaching

In Grade 9, students solved problems involving operations on rational numbers (9N3). This will now be extended to multiplying, dividing, adding, and subtracting rational expressions with numerators and denominators limited to monomials, binomials and trinomials.

Multiplying and dividing rational expressions is very similar to the process students used to multiply and divide rational numbers. Using examples such as $\frac{12}{25}\times\frac{10}{21}$ and $\frac{x^2-9}{x^2-4x}\times\frac{x-4}{x-3}$, ask students to simplify and find the product for each. They should think about whether the strategy for multiplying rational expressions is the same as the strategy for multiplying rational numbers. Ask them to also consider at what step the non-permissible values are determined.

It is important for students to recognize the importance of factoring the numerator and denominator of the rational expression, if possible, before the product is determined. Ask students to answer the following:

- Find the product of $\frac{x^2}{x^2-4}\times\frac{x+2}{x}$ using two different strategies. Which strategy is more efficient? Why?

Reinforce that multiplication of rational expressions follows the same procedure as multiplying rational numbers, but with the added necessity of determining the non-permissible values for the variables.

Provide an opportunity for students to compare the division of rational numbers to division of rational expressions. Students sometimes forget to identify the non-permissible values for the numerator of the divisor in a division statement. Reinforce the importance of this step through the use of examples.

Students should recognize that a multiplication of two or more rational expressions can be written as a single expression. For example,

$$\frac{x(x-2)}{x+3}\times\frac{2x-1}{x(x+2)}$$
$$=\frac{x(x-2)(2x-1)}{x(x+3)(x+2)}$$

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to create an activity sheet where the column on the left contains operations with rational expressions and the column on the right contains the non-permissible values (not in the same order). Students will then exchange their sheets. The task is to match each expression with its correct non-permissible values. (the non-permissible values may match more than one expression on the left and may not match any).

(AN5.2)

- Ask students to work in groups to complete the following table. Students should explain the similarities between finding the lowest common denominator (LCD) of two rational numbers versus two rational expressions.

Rational Number	LCD	Rational Expression	LCD	Similarities
$\frac{4}{5}+\frac{3}{5}$		$\frac{6}{2x-1}+\frac{-2}{2x-1}$		
$\frac{1}{5}-\frac{7}{15}$		$\frac{4x}{x-3}-\frac{5}{6x-18}$		
$\frac{7}{12}+\frac{3}{8}$		$\frac{2}{x^2-36}+\frac{4}{3x+18}$		

(AN5.1)

Journal

- Nigel stated that the permissible values for the quotient and the product of the expressions $\frac{x^2+4x+3}{x^2-16}$ and $\frac{x^2-2x-8}{x^2-7x+12}$ are the same. Ask students if they agree or disagree with his statement. They should justify their answer.

(AN5.2, AN5.3)

Resources/Notes

Authorized Resource

Pre-Calculus 11

6.2 Multiplying and Dividing Rational Expressions

SB: pp. 322-330
TR: pp. 217-223
BLM: 6-4, 6-6

Algebra and Number

Outcomes

Students will be expected to

AN5 Continued ...

Achievement Indicators:

AN5.4 *Determine, in simplified form, the sum or difference of rational expressions with the same denominator.*

AN5.5 *Determine, in simplified form, the sum or difference of rational expressions in which the denominators are not the same and which may or may not contain common factors.*

AN5.2 *Continued*

Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students were introduced to the lowest common multiple for a set of numbers (AN1.3). Students should compare finding the lowest common denominator of rational numbers to finding the lowest common denominator of rational expressions. Allow students to discover the different situations that occur when finding the lowest common denominator of two fractions and then compare this to rational expressions. Consider the following table:

Rational Number	Situation	Rational Expression
$\frac{3}{7}-\frac{2}{7}$	the denominators are the same	$\frac{x^2}{x+1}-\frac{1}{x+1}$
$\frac{1}{12}+\frac{5}{6}$	one denominator is a multiple of the other	$\frac{3}{x+5}-\frac{1}{4x+20}$
$\frac{2}{3}+\frac{7}{2}$	the denominators have no common factors	$\frac{3}{2x}+\frac{4}{x-1}$
$\frac{5}{14}+\frac{1}{6}$	the denominators have a common factor	$\frac{7}{x^2-9}+\frac{1}{4x+12}$

Ask students to answer the following questions related to the rational expressions:

- How do you find the lowest common denominator? Why is it beneficial to simplify the expression before finding the lowest common denominator?
- What are the non-permissible values?
- Can you list other examples that fit each situation?

Similar to rational numbers, rational expressions can be added if they have common denominators. Once students determine the lowest common denominator, they should rewrite each rational expression with that common denominator.

A common student error involves adding or subtracting the numerators without first writing the fractions with a common denominator. For example, students mistakenly add $\frac{x}{5}+\frac{2}{3}$ as $\frac{x+2}{8}$. Remind students to be careful when subtracting rational expressions. They sometimes forget to distribute the negative sign when there is more than one term in the numerator. For example, $\frac{3x-2}{(x+2)(x-2)}-\frac{2x-4}{(x+2)(x-2)}$ is often written as $\frac{3x-2-2x-4}{(x+2)(x-2)}$. Encourage students to use brackets to help them avoid this mistake.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to simplify the rational expressions:

  (i) $\frac{x+7}{2x+14} - \frac{5x}{-3x-21}$

  (ii) $\frac{2x-6}{x^2-x-6} - \frac{3x+12}{x^2+x-12}$

(AN5.5)

Resources/Notes

Authorized Resource

Pre-Calculus 11

6.3 Adding and Subtracting Rational Expressions

SB: pp. 331-340
TR: pp. 224-231
BLM: 6-4, 6-7

Algebra and Number

Outcomes

Students will be expected to

AN5 Continued ...

Achievement Indicator:

AN5.6 *Simplify an expression that involves two or more operations on rational expressions.*

Elaborations—Strategies for Learning and Teaching

A complex fraction is a fraction where the numerator, denominator, or both, contain one or more fractions. In order to avoid errors, students should place brackets appropriately and use the order of operations correctly.

Provide examples to students illustrating various strategies to simplify an expression containing a complex fraction. Students may first simplify both the numerator and denominator, invert and multiply and then simplify. Consider the example :

$$\frac{\frac{1}{x+2}+\frac{1}{x-2}}{\frac{x}{x^2-4}+\frac{1}{x+2}}=\frac{\frac{x-2}{(x+2)(x-2)}+\frac{x+2}{(x+2)(x-2)}}{\frac{x}{(x+2)(x-2)}+\frac{x-2}{(x+2)(x-2)}}=\frac{2x}{(x+2)(x-2)}\times\frac{(x+2)(x-2)}{2x-2}=\frac{2x}{2x-2}=\frac{x}{x-1}$$

As an alternative, students can multiply the entire expression by the common denominator divided by itself. This common denominator is obtained by considering all existing denominators in the expression.

$$\left(\frac{\frac{1}{x+2}+\frac{1}{x-2}}{\frac{x}{x^2-4}+\frac{1}{x+2}}\right)\frac{(x+2)(x-2)}{(x+2)(x-2)}=\frac{\frac{1}{x+2}(x+2)(x-2)+\frac{1}{x-2}(x+2)(x-2)}{\frac{x}{(x-2)(x+2)}(x+2)(x-2)+\frac{1}{x+2}(x+2)(x-2)}$$

$$=\frac{x-2+x+2}{x+x-2}=\frac{2x}{2x-2}=\frac{x}{x-1}$$

As students work through the two possible strategies, they should think about efficiency and what will work best for any complex fraction.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

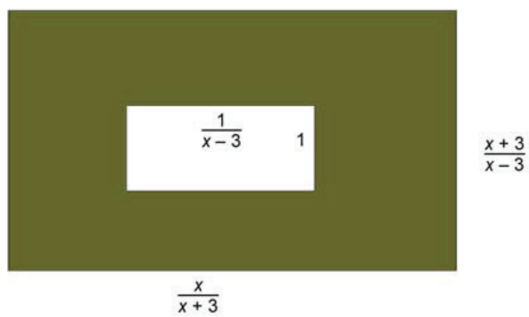
Presentation

- Ask students, working in pairs or small groups, to create two rational expressions. The first rational expression should contain each of the operations. The other expression would involve two or more operations on rational expressions. They should solve the expressions and present their findings to the class.

(AN5.3, AN5.4, AN5.5, AN5.6)

Paper and Pencil

- Ask students to generate an expression representing the area of the shaded region.



(AN5.6)

Resources/Notes

Authorized Resource

Pre-Calculus 11

6.3 Adding and Subtracting Rational Expressions

SB: pp. 331-340
TR: pp. 224-231
BLM: 6-4, 6-7

Algebra and Number

Outcomes

Students will be expected to

AN6 Solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials or trinomials).

[C, PS, R]

Achievement Indicators:

AN6.1 *Determine the non-permissible values for the variable in a rational equation.*

AN6.2 *Determine the solution to a rational equation algebraically, and explain the strategy used to solve the equation.*

AN6.3 *Explain why a value obtained in solving a rational equation may not be a solution of the equation.*

Elaborations—Strategies for Learning and Teaching

In Grade 9, students solved linear equations (9PR3). Previously in this course, students solved quadratic equations and identified inadmissible roots (RF5). They will now solve equations containing rational expressions and check if the solutions are permissible values. Roots that are non-permissible are extraneous. (RF2). It is intended that the rational equations be those that can be simplified to linear and quadratic equations.

Students will be exposed to different strategies when solving rational equations. It would be beneficial to begin with an example that is easier to visualize before moving on to more complex equations. Some students may use trial and error to solve an equation such as $\frac{x}{10} = \frac{2}{5}$. Others may be able to determine the solution by inspection. Encourage students to discuss their ideas. For example, a student may respond that in order to get the number 10, the number 5 must be doubled therefore 2 is also doubled resulting in $x = 4$. This student response is a great lead into the strategy of creating an equivalent rational equation with common denominators. Ask students to rewrite the rational equation with a common denominator ($\frac{x}{10} = \frac{4}{10}$) and then write an equation with the numerators.

Another strategy involves eliminating the denominators. Use an example such as, $\frac{x}{10} = \frac{2}{5}$, to promote discussion around lowest common denominator:

- What is the lowest common denominator of 10 and 5?
- What would happen if the lowest common denominator was multiplied on both sides of the equation? Why is this mathematically correct?
- What is the simplified equation?
- What is the solution?

Once the strategies have been discussed, students will then be exposed to solving more complex rational equations.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Paper and Pencil

- Provide each student with a rational equation on a sheet of paper. Ask them to identify the non-permissible roots and then solve the equation. Ask students not to write their name on the paper since student solutions will be collected and then redistributed randomly around the room. Ask students to verify if the solution is correct. If the solution is incorrect, they will identify the error and write the correct solution.

(AN6.1, AN6.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11

6.4 Rational Equations

SB: pp. 341-351
TR: pp. 232-238
BLM: 6-4, 6-8

Algebra and Number

Outcomes

Students will be expected to

AN6 Continued ...

Achievement Indicators:

AN6.1, AN6.2

AN6.3 *Continued*

Elaborations—Strategies for Learning and Teaching

When solving $\frac{3}{x}+\frac{7}{2x}=\frac{1}{5}$, for example, students may find a common denominator for the left side of the equation and then proceed to solve $\frac{13}{2x}=\frac{1}{5}$. They can also multiply both sides of the equation by the lowest common denominator $(10x)\frac{3}{x}+(10x)\frac{7}{2x}=(10x)\frac{1}{5}$. It is important, however, for students to recognize that they can reduce the numbers of steps by multiplying both sides of the original equation by the lowest common denominator.

Students can add or subtract the terms on the left hand side or the right hand side of the equation before they cross multiply. This process, however, may lead to an equation where the degree of the polynomial is greater than what they started with. Consider the following example:

$$\frac{2x^2+1}{x+3}=\frac{x}{4}+\frac{5}{x+3}$$

$$\frac{2x^2+1}{x+3}=\frac{x^2+3x+20}{4(x+3)}$$

$$4(2x^2+1)(x+3)=(x^2+3x+20)(x+3)$$

This example results in a cubic equation. Students are only familiar with solving quadratic equations at this point. Therefore, multiplying both sides of the equation by the lowest common denominator would be the method students would choose.

Caution students that it is necessary to find the non-permissible roots at the beginning of the solution since some rational equations may lead to extraneous roots. Consider the equation $\frac{2x+3}{x+5}+\frac{1}{2}=\frac{-14}{2(x+5)}$. Ask students to answer the following questions:

- What is the non-permissible root? What does this mean?
- What is the solution to the resulting linear equation?
- Why is it important to check the solution by using the original equation?

Students should recognize solutions that are non-permissible values are extraneous roots. Therefore, they must be eliminated as a valid solution.

When solving rational equations, the modified equation may result in either a linear or quadratic equation. Students will have a choice whether to use the quadratic formula or their factoring skills to solve the quadratic equation. Remind them to verify their solutions to avoid extraneous roots.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Journal

- Ask students to reflect on the process of solving a rational equation. They should respond in writing to three reflective prompts providing six responses, as shown below, to describe what they learned.

3 things I understand:
1.
2.
3.
2 things I am still struggling with:
1.
2.
1 thing that I will work on:
1.

(AN6.1, AN6.2, AN6.3)

Paper and Pencil

- Ask students to solve and verify:

  (i) $\frac{x}{4} - \frac{7}{x} = 3$

  (ii) $\frac{3x}{x+2} - \frac{5}{x-3} = \frac{-25}{x^2 - x - 6}$

(AN6.1, AN6.2, AN6.3)

Resources/Notes

Authorized Resource

Pre-Calculus 11

6.4 Rational Equations

SB: pp. 341-351
TR: pp. 232-238
BLM: 6-4, 6-8

Algebra and Number

Outcomes

Students will be expected to

AN6 Continued ...

Achievement Indicator:

AN6.4 *Solve problems by modeling a situation using a rational equation.*

Elaborations—Strategies for Learning and Teaching

Students will be expected to write an equation to represent a problem. They should be exposed to examples such as the following:

The sum of a number and its reciprocal is $\frac{5}{2}$. Students can begin this example using trial and error and discuss possible solutions. They should then proceed to write the rational equation $x + \frac{1}{x} = \frac{5}{2}$. Encourage students to write their own example and share with the class.

It is important for students to recognize that inadmissible roots come from the context of the problem. Discuss different scenarios that produce inadmissible roots. A negative numerical value, for example, would not make sense if referring to time, height and length.

Students may have difficulty interpreting the information from the word problem and writing the rational equation. Encourage them to use tables and diagrams to help them break down the information. Consider the following example:

Sherry mows a lawn in 4 hours. Mary mows the same lawn in 5 hours. How long would it take both of them working together to mow the lawn? Pose the following questions to begin a discussion:

- How much of the lawn would Sherry mow in 1 hour?
- How much of the lawn would Mary mow in 1 hour?
- How much of the lawn would both mow together in 1 hour?

Completing a table such as the one below should help students organize their information.

	Time to mow lawn (hours)	Fraction of lawn mowed in 1 hr
Sherry	4	$\frac{1}{4}$
Mary	5	$\frac{1}{5}$
Both	x	$\frac{1}{x}$

When solving the equation $\frac{1}{4} + \frac{1}{5} = \frac{1}{x}$, encourage students to check that the solutions satisfy the original equation are permissible, and in the case of a word problem, realistic in the context.

Students should also be exposed to word problems that produce a rational equation resulting in solving a quadratic equation. In such cases, there may be an inadmissible value that will need to be rejected in the context of the problem.

General Outcome: Develop algebraic reasoning and number sense.

Suggested Assessment Strategies

Performance

- Create pairs of cards with word problems and matching equations to solve the word problems. Distribute the cards amongst the students and have them find their partner by matching the word problem with the corresponding equation. Once they have found their partner, students should work in pairs to solve the equation and verify their solution.

(AN6.2, AN6.4)

Paper and Pencil

- A student was given the following word problem:

  It takes Mike 9 hours longer to construct a fence than it takes Katya. If they work together, they can construct the fence in 20 hours. How long would it take Mike to construct the fence alone?

  The student solved the equation $\frac{20}{t}+\frac{20}{t+9}=1$ and stated the solutions to the word problem were 36 and -5. Ask students to verify the solution and state whether the student is correct.

(AN6.2, AN6.3, AN6.4)

- Ask students to solve:

  A boat goes downstream from Port A to Port B in 4 hours. Maintaining the same speed, it covers the same distance upstream in 5 hours. If the speed of the water is 2 km/h, find the speed of the boat.

(AN6.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11

6.4 Rational Equations

SB: pp. 341-351
TR: pp. 232-238
BLM: 6-4, 6-8

Absolute Value and Reciprocal Functions

Suggested Time: 12 Hours

Unit Overview

Focus and Context

In this unit, students will focus on absolute values, determine the absolute value of numerical expressions, and solve problems involving the absolute value. They will graph the absolute value of a linear function and the absolute value of a quadratic function. Students will identify characteristics of the graph, including the intercepts and the domain and range. They will also represent the absolute value function using piecewise notation. Students will solve absolute value equations graphically and algebraically.

Students will explore reciprocal functions by comparing the graphs of a function and its reciprocal. The focus will be on linear and quadratic functions.

Outcomes Framework

GCO

Develop algebraic reasoning and number sense.

- **SCO AN1**

  Demonstrate an understanding of the absolute value of real numbers.

GCO

Develop algebraic and graphical reasoning through the study of relations.

- **SCO RF2**

  Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems.

- **SCO RF11**

  Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions).

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
	Algebra and Number	
	AN1 Demonstrate an understanding of the absolute value of real numbers. [R, V]	
Relations and Functions	Relations and Functions	Relations and Functions
RF5 Determine the characteristics of the graphs of linear relations, including the: • intercepts • slope • domain • range [CN, PS, R, V] RF6 Relate linear equations expressed in: • slope-intercept form ($y = mx + b$) • general form ($Ax + By + C = 0$) • slope-point form ($y - y_1 = m(x - x_1)$) to their graphs. [CN, R, T, V]	RF2 Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. [C, PS, R, T, V] RF11 Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). [CN, R, T, V]	

Mathematical Processes

[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

Algebra and Number

Outcomes

Students will be expected to

AN1 Demonstrate an understanding of the absolute value of real numbers.
[R, V]

Achievement Indicators:

AN1.1 *Determine the distance of two real numbers of the form ±a, $a \in \mathbb{R}$, from 0 on a number line, and relate this to the absolute value of a ($|a|$).*

AN1.2 *Determine the absolute value of a positive or negative real number.*

AN1.3 *Explain, using examples, how distance between two points on a number line can be expressed in terms of absolute value.*

AN1.4 *Determine the absolute value of a numerical expression.*

AN1.5 *Compare and order the absolute values of real numbers in a given set.*

Elaborations—Strategies for Learning and Teaching

The concept of absolute value is new to students. They will determine the absolute value of numerical expressions and compare and order the absolute values of real numbers in a given set.

Introduce students to the absolute value of a number as its distance from zero. Ask them to plot the integers 5 and -5 on a number line. They should notice, when graphed, that these numbers are the same distance from zero. It is important for students to understand that absolute value only asks “how far?”, not “in which direction?”. Reinforce that distance is always positive. This leads to the definition of the absolute value of any real number *a*:

$$|a| = \begin{cases} a, \text{ if } a \geq 0 \\ -a, \text{ if } a < 0 \end{cases}$$

Students will extend the concept of absolute value to include the distance between any two real numbers. This could be first investigated using natural numbers and then extended to integers and real numbers. Completing a table such a the one below should help students recognize that the distance between *a* and *b* can be represented by $|a - b|$ or $|b - a|$.

a	*b*	distance between *a* and *b*	value of $\|a - b\|$	value of $\|b - a\|$
2	6	4	$\|2-6\|=\|-4\|=4$	$\|6-2\|=\|4\|=4$
-5	-10			
2.68	5.75			

In Grade 9, students simplified numerical expressions using the order of operations (9N4). This is now extended to include expressions containing absolute value. Ask students to compare the expressions 3 - 4(2) and |3 - 4(2)|. They may have difficulty determining where the absolute value of a numerical expression occurs in the order of operations. They will simplify the expression inside the absolute value symbol using the order of operations and then take the absolute value of the resulting expression.

Students will compare and order absolute values of real numbers. If the absolute value contains a fraction, students may find it helpful to change it to a decimal representation. Once students evaluate and compare the absolute values in a given set, they can place the values on a number line to help them order the set.

General Outcome: Develop algebraic reasoning and number sense

Suggested Assessment Strategies

Journal

- Provide students a fictitious town with one main street. This street should contain at least eight landmarks located to the right and left of the town square which is representative of the origin. Each landmark should be given a name (gas station, library, etc.) Ask students to pose three questions where the distance between any two landmarks is requested. They should then answer the questions.

(AN1.3)

Observation

- Create a set of cards with a variety of examples involving the absolute values of real numbers and numerical expressions. In groups of three, each student will receive five cards. They will place the cards in position that produces an ordered set of cards. When each member of the group is finished, ask each student to explain their reasoning to the other members.

(AN1.2, AN1.4, AN1.5)

- Ask students to create a human number line. Each student is given a card containing an absolute value of a real number or numerical expression. They order themselves into a line based on the relative size of their number. Ask them to explain why they chose their position. As an alternative, a skipping rope or piece of string could be used. Students attach their number to the line in the appropriate position.

(AN1.2, AN1.4, AN1.5)

Performance

- Divide the class into teams consisting of about five students. Each team would line up one behind the other. Provide each team with a list of absolute value problems to simplify. The first student in each line would write the answer to one of the absolute value problems using mental math skills. They would then move to the back of the line and the second student would repeat this with a different problem. The first team to complete their questions with the correct answers would be the winners.

(AN1.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11

7.1 Absolute Value

Student Book (SB): pp. 358-367
Teacher Resource (TR): pp. 250-255
Blackline Master (BLM): 7-3, 7-4

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/absval.html

- interactive game for ordering the absolute values of real numbers

Relations and Functions

Outcomes

Students will be expected to

RF2 Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems.

[C, PS, R, T, V]

Achievement Indicators:

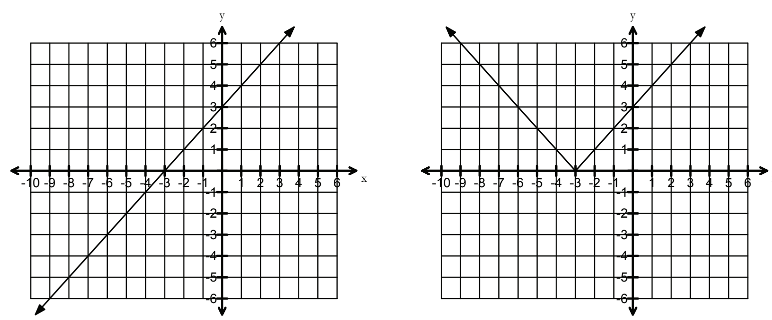
RF2.1 *Create a table of values for $y = |f(x)|$, given a table of values for $y = f(x)$.*

RF2.2 *Sketch the graph of $y = |f(x)|$; state the intercepts, domain and range; and explain the strategy used.*

Elaborations—Strategies for Learning and Teaching

In Grade 9, students solved linear equations (9PR3). In Mathematics 1201, students analyzed the graphs of linear relations (RF6). Earlier in this course, students analyzed quadratic functions in vertex and standard form to identify the characteristics of the corresponding graph (RF3, RF4). They will now graph and analyze absolute value functions, limited to linear and quadratic functions.

Students will compare the graph of a linear function to its corresponding absolute value function, using a table of values. They should identify the similarities and differences between $y = |ax + b|$ and $y = ax + b$. Ask students to graph $y = |x + 3|$, for example, using the table of values for $y = x + 3$. To get an accurate picture of the absolute value graph, it is important to include appropriate x-values that will produce negative y-values.



Using the graph as a visual aid, students should recognize that the section of the graph which lies above the x-axis remains the same and the section which lies below the x-axis is reflected in the x-axis.

Ask students to identify characteristics of the absolute value graph, such as the intercepts, domain and range. As they look for the similarities and differences between the graphs, they should consider the following:

- Is the x-intercept significant?
- Is the domain dependent on the x-intercept? Explain.
- Why is the domain of the the function the same as the domain of the absolute value function? Why is the range different?

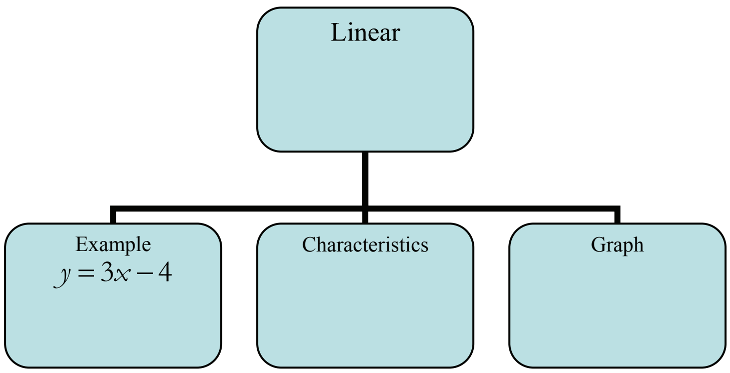
Comparing the graphs of $y = |ax + b|$ and $y = ax + b,$ students recognize the resulting shape of the graph. Using this shape, along with the x-and y-intercepts, they should be able to draw the graph.

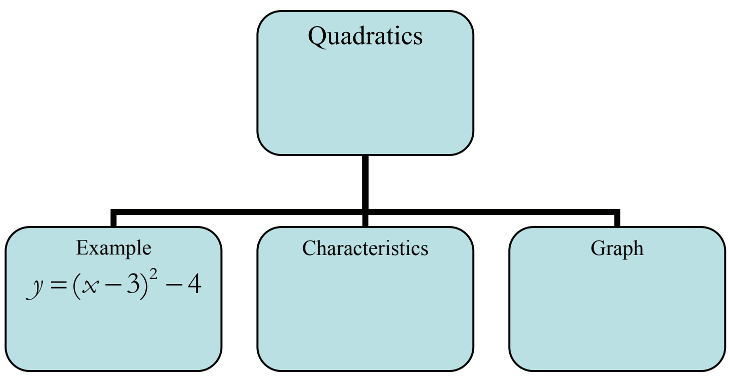
General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- As a pre-assessment, ask students to complete the following flowchart to describe what they know about linear and quadratic functions.





(RF2.1, RF2.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11

7.2 Absolute Value Functions

SB: pp. 368-379
TR: pp. 256-261
BLM: 7-3, 7-5

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/absval.html

- pre-assessment template

Relations and Functions

Outcomes

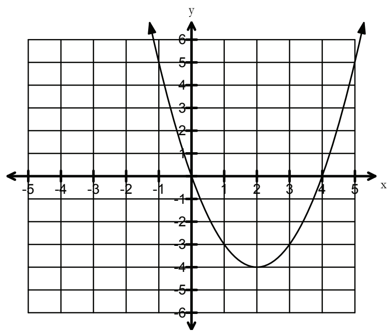
Students will be expected to

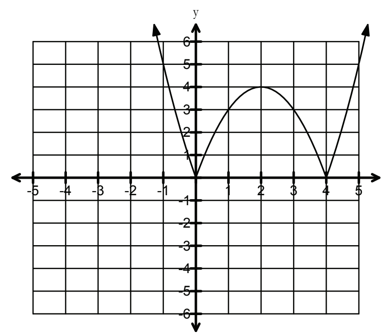
RF2 Continued ...
Achievement Indicators:

RF2.1, RF2.2 *Continued*

Elaborations—Strategies for Learning and Teaching

Students will then proceed to work with and analyze the absolute value of quadratic functions. They will compare the graphs of the absolute value of a quadratic function to its original graph using a table of values. Ask students to graph, for example, $y = |(x - 2)^2 - 4|$ using the table of values for $y = (x - 2)^2 - 4$. Remind them that the graph of a quadratic function can be obtained using vertex and the x-intercepts, as previously introduced in the Quadratic Functions unit.

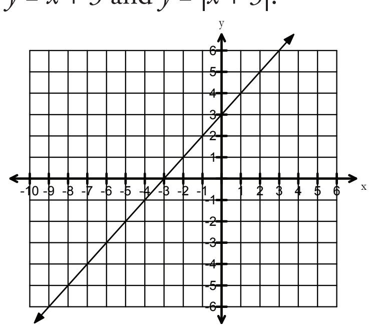


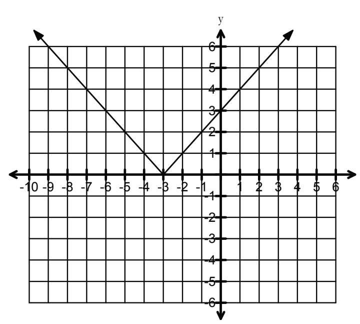


Students should notice that the graph of $y = |f(x)|$ reflects the negative part of the graph of $y = f(x)$ across the x-axis, while the positive part remains unchanged. Students should again note the similarities and differences between the two graphs.

RF2.3 *Generalize a rule for writing absolute value functions in piecewise notation.*

Students will develop a rule for writing absolute value functions in piecewise notation using a graph. Explain to students that piecewise functions are used to describe functions that contain distinct functions over different intervals. The graph of a linear absolute value function, for example, consists of two separate linear functions. The domain for each interval is dependent on the x-intercept(s). Revisit the graphs of $y = x + 3$ and $y = |x + 3|$.





Students should notice the turning point on the absolute value graph is located at the x-intercept. For values of x where y is positive, the graphs of $y = |x + 3|$ and $y = x + 3$ are the same. For values of x where y is negative, the graph of $y = |x + 3|$ is a reflection of the graph of $y = x + 3$.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Performance

- Ask students to work in groups for this activity. Each group will be given a different tables of values. They will create a table of values for $y = |f(x)|$. Ask students to display their tables around the classroom. They should observe and analyze the data. Each group will then present their conclusions to the rest of the class.

(RF2.1)

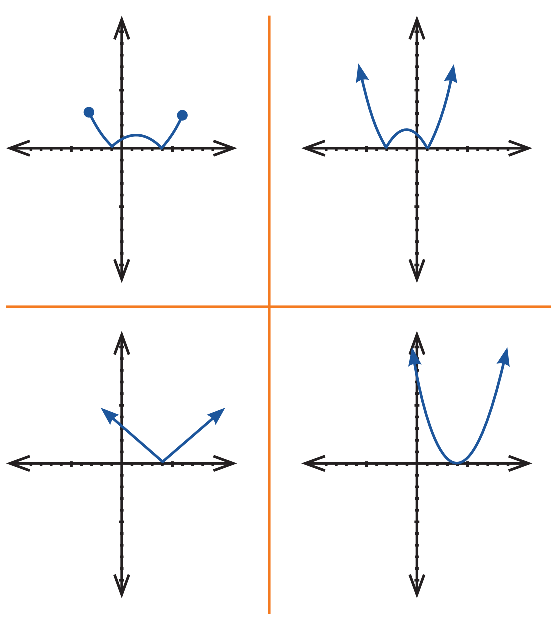
- Working in small groups, students will play the *Absolute Matching Game*. Each group will be given several cards. Half of the cards will have the graph of $y = f(x)$ while the other half will have the graph of $y = |f(x)|$. The object of the game is to be the first group to pair up each graph with its absolute value graph.

  Note: For this game to be most effective, the graphs should have similar characteristics, such as the same x-intercepts but different slopes/vertices.

(RF2.2)

Observation

- Invite students to play WODB (Which One Doesn't Belong?). Show students the following graphs. Each graph could be the one which doesn't belong, but for a different reason. Observe students' reasoning for misconceptions.



(RF2.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11

7.2 Absolute Value Functions

SB: pp. 368-379
TR: pp. 256-261
BLM: 7-3, 7-5

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/absval.html

- *Absolute Matching Game*
- WODB graphs

Relations and Functions

Outcomes

Students will be expected to

RF2 Continued ...

Achievement Indicator:

RF2.3 *Continued*

Elaborations—Strategies for Learning and Teaching

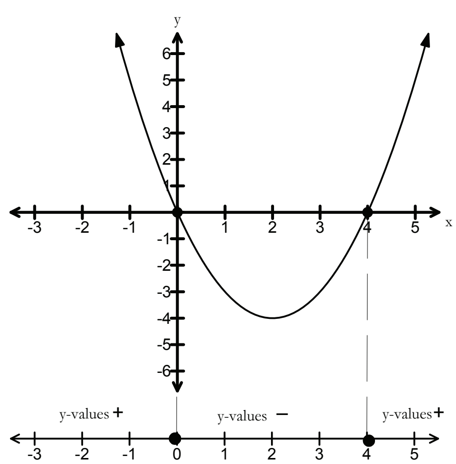
The absolute value function $y = |x + 3|$ can be written in piecewise notation as:

$$y = \begin{cases} x + 3, \textit{ if } x \geq -3 \\ -(x+3), \textit{ if } x < -3 \end{cases}$$

This can be generalized to all absolute value functions:

$$y = \begin{cases} f(x), \text{ where } f(x) \geq 0 \\ -f(x), \text{ where } f(x) < 0 \end{cases}$$

Similarly, students can use the visual representation of the absolute value of a quadratic function to determine the piecewise function. This would be a good opportunity to also expose students to an algebraic method using sign diagrams to analyze where the quadratic function is positive or negative. Ask students to write, for example, the function $y = |(x - 2)^2 - 4|$ using piecewise notation. They will first use the quadratic function $y = (x - 2)^2 - 4$ to determine the x-intercepts. They will then determine the distinct intervals where the function is positive and where it is negative on a number line.



Using the sign diagram, students can write the piecewise function as:

$$y = \begin{cases} (x-2)^2 - 4, \ \ x \leq 0 \text{ or } x \geq 4 \\ -[(x-2)^2 - 4], \ \ 0 < x < 4 \end{cases}$$

This can also be done without the graph as a visual aid. Relating the sign diagram to the x-axis of the graph, students could substitute an x-value from each interval into the function to determine where the function is positive or negative.

When graphing the absolute value of a linear or quadratic function, they should recognize the shape of the graph, the y-intercept and the x-intercept(s). Using the information from the sign diagram, they can then draw the graph of the absolute value function.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Performance

- In small groups, ask students to participate in a card game, *Piecewise Pairs*, involving graphs and piecewise functions. Each group will be given several cards. Half the cards contain a graph while the other half will have a piecewise function. The object of the game is to be the first group to match each graph with its correct piecewise notation.

  Note: Graphs should be similar enough to each other so that students cannot just match them without any thought (e.g., have several graphs with the same *x*-intercept so additional analysis is involved).

(RF2.3)

Paper and Pencil

- Ask students to sketch the graph of $y = \left|-x^2 + 2x + 3\right|$ and state as a piecewise function.

(RF2.2, RF2.3)

Resources/Notes

Authorized Resource

Pre-Calculus 11

7.2 Absolute Value Functions

SB: pp. 368-379
TR: pp. 256-261
BLM: 7-3, 7-5

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/absval.html

- *Piecewise Pairs*

Relations and Functions

Outcomes

Students will be expected to

RF2 Continued...

Achievement Indicator:

RF2.4 *Solve an absolute value equation graphically, with or without technology.*

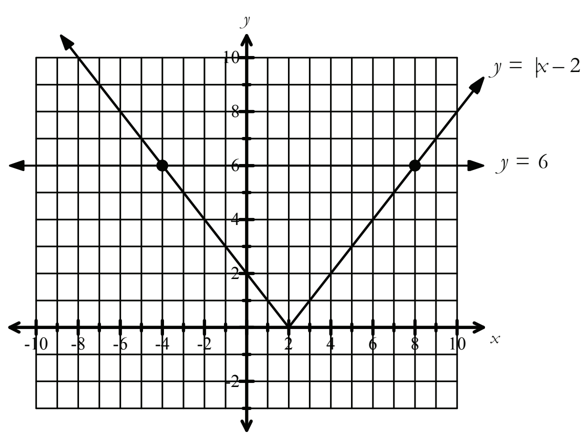
Elaborations—Strategies for Learning and Teaching

Students will explore the solutions to absolute value equations, focusing first on the graphical representation and then moving to an algebraic solution. Remind them of the properties of absolute value:

- For a real number a, the absolute value $|a|$ is the distance from a to the origin.
- For two real numbers a and b, $|a - b|$ is the distance between a and b on the number line.

Students should first work with the absolute value of a linear equation before moving on to the absolute value of a quadratic equation. Ask students what it means to solve an equation such as $|x - 2| = 6$. They should be looking for points whose distance from 2 is 6. Using a number line, they should realize that both -4 and 8 are at a distance of 6 from 2. This reasoning will allow students to better understand the solutions when using a graph.

Ask students to graph the function $y = |x - 2|$ and $y = 6$ on the same coordinate plane to determine the points of intersection.



They should notice the x-coordinates of the points of intersection are the solutions to the equation. Encourage students to verify the solution by substituting the values back into the equation. Similarly, this graphical approach can be used to solve the absolute value of quadratic equations.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to:

  (i) Graph $y=|x^2-2|$ and $y=2$ on the same set of axes.

  (ii) Identify the solutions to the equation $2=|x^2-2|$.

  (iii) Verify the solutions algebraically.

(RF2.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11

7.2 Absolute Value Functions

SB: pp. 368-379
TR: pp. 256-261
BLM: 7-3, 7-5

7.3 Absolute Value Equations

SB: pp. 380-391
TR: pp. 262-267
BLM: 7-3, 7-6

Relations and Functions

Outcomes

Students will be expected to

RF2 Continued...

Achievement Indicators:

RF2.5 *Solve, algebraically, an equation with a single absolute value, and verify the solution.*

RF2.6 *Explain why the absolute value equation $|f(x)| < 0$ has no solution.*

RF2.7 *Determine and correct errors in a solution to an absolute value equation.*

RF2.8 *Solve a problem that involves an absolute value function.*

Elaborations—Strategies for Learning and Teaching

Students will move from the graphical representation of a solution to an algebraic method to solve absolute value equations.

Remind students that the definition of an absolute value consists of two parts: where $f(x) \geq 0$ and $f(x) < 0$. Consider the previous example $|x - 2| = 6$. The equation is split into two possible cases: $\pm (x - 2) = 6$. Students will solve the equations $(x - 2) = 6$ and $-(x - 2) = 6$ to result in $x = -4$ and $x = 8$. Each solution should be checked for extraneous roots.

When students solve absolute value equations involving a quadratic expression, such as $|x^2 - 4| = 3x$, they may have to solve the resulting quadratic equation using their factoring skills or the quadratic formula. Encourage them to verify the solutions by substituting the x-values back into the equation and making sure the left hand side of the equation is equal to the right hand side of the equation.

Remind students the absolute value of a number is always greater than or equal to zero. Students can use inspection to see if this property applies. Consider the example $|x - 2| = -6$. Ask students if their is any possible x value that can be substituted into the equation to result in a negative value.

Students should be provided with worked solutions containing errors to a number of absolute value equations and asked to identify and correct the errors. Common errors include:

- Treating the absolute value sign like parentheses, for example, multiplying a constant by the expression within the absolute value sign (e.g., $-2|x - 3| = |-2x + 6|$).
- Incorrectly placing the negative in front of the variable rather than the entire expression (i.e., When solving $|x - 3| = 8$, students may write $-x - 3 = 8$ instead of $-(x - 3) = 8$).
- Not identifying extraneous roots.
- Errors when using the quadratic formula.

Problem solving using linear absolute value functions should be embedded throughout the unit and situated in a variety of contexts. In some problems, students will be given an absolute value function and asked to analyze it. In other problems, students will be required to create the function from the given information. Encourage students to check their answers and identify extraneous solutions.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to illustrate with a graph why $|x - 2| = -6$ has no solution.

(RF2.6)

- Place cards on the classroom wall consisting of linear and quadratic absolute value equations. Ask students to choose two cards, one linear and one quadratic. They should solve the equations and pass them in as exit cards. These exit cards can then be used later for students to determine and correct any errors.

(RF2.5, RF2.7)

- For the activity *Pass the Problem*, each group of students is given absolute value equations to solve algebraically. After a specific amount of time, ask students to swap their problem with another pair. If the group finished the problem, the other group will check the solution. If errors are identified the group will correct the error and then continue to complete the problem. If the group, however, did not finish answering the problem, the other pair will check the partially completed solution and pick up from where the group left off. When they are finished, they should share the completed responses with each other, defending their reasons for any changes they made and provide feedback on each other's thinking.

(RF2.5, RF2.7)

Presentation

- Ask groups of students to research a problem that incorporates the absolute value function. They should present the problem to the class, including the solution.

(RF2.8)

Resources/Notes

Authorized Resource

Pre-Calculus 11
7.3 Absolute Value Equations

SB: pp. 380-391
TR: pp. 262-267
BLM: 7-3, 7-6

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/absval.html

- absolute value equation game

Relations and Functions

Outcomes

Students will be expected to

RF11 Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions).

[CN, R, T, V]

Achievement Indicators:

RF11.1 *Compare the graph of* $y = \frac{1}{f(x)}$ *to the graph of* $y = f(x)$.

RF11.2 *Identify, given a function f(x), values of x for which* $y = \frac{1}{f(x)}$ *will have vertical asymptotes; and describe their relationship to the non-permissible values of the related rational expression.*

RF11.3 *Graph, with or without technology,* $y = \frac{1}{f(x)}$ *, given* $y = f(x)$ *as a function or a graph, and explain the strategies used.*

Elaborations—Strategies for Learning and Teaching

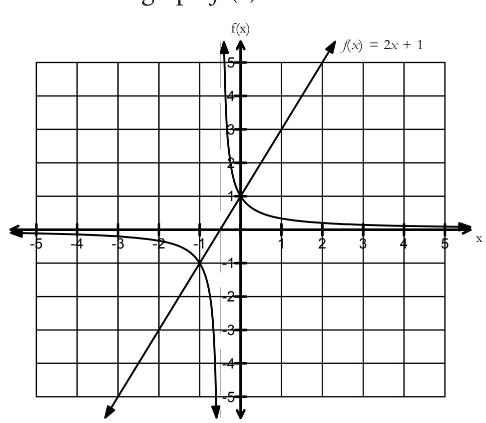
Students will compare the graphs of a function and its reciprocal. Examples will include both linear and quadratic functions.

When graphing a function and its reciprocal, there are invariant points that remain unchanged. Consider showing students the following table to highlight the invariant points using a function such as $f(x) = 2x + 1$.

x	$f(x)$	$\frac{1}{f(x)}$
1	3	$\frac{1}{3}$
0.5	2	$\frac{1}{2}$
0	1	1
-0.5	0	undefined
-1	-1	-1

Students should notice the invariant points can be located by setting $f(x) = \pm 1$. These points will help students when they graph reciprocal functions, as these will be common to both functions and are therefore the points of intersection.

As students explore linear and quadratic reciprocal functions, there are certain characteristics that should be addressed in a class discussion. Ask students to graph $f(x) = 2x + 1$.



General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Presentation

- Ask students to use graphing technology to analyze and describe the characteristics of a function and its reciprocal, and vice versa. As teachers observe this activity they should ask questions such as: Where do the graphs intersect? What do the vertical asymptotes of the reciprocal function correspond to in the original function?

  (RF11.1, RF11.2, RF11.3, RF11.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11

7.4 Reciprocal Functions

SB: pp. 392-409
TR: pp. 268-276
BLM: 7-3, 7-7

Relations and Functions

Outcomes

Students will be expected to

RF11 Continued...

Achievement Indicators:

RF11.1, RF11.2

RF11.3 *Continued*

Elaborations—Strategies for Learning and Teaching

Students should consider the following:

- If the point (x, y) is on the graph $y = f(x)$, what do you notice about the point on the graph of the reciprocal function?
- Is the sign of the reciprocal function the same as the sign of the original function?
- What do you notice about the x-intercept of the linear function, the non-permissible value of the reciprocal function, and the location of the vertical asymptote?
- What is the horizontal asymptote?
- What are the invariant points?
- What do you notice about the behaviour of the reciprocal function as it approaches the asymptotes?

They should also recognize, from the table and the graph, that the x-intercept of the function produces a non-permissible value for the reciprocal function. This is the location of the vertical asymptote. When analyzing and comparing the graphs of the function and its reciprocal, it is important to note that as the y-values of one function increase the y-values of the other decrease and vice versa. This could be demonstrated by looking at parts of the graph on either side of the invariant points.

As students recognize the shape of the graph along with the intercepts and the asymptotes, they should be able to sketch the graph of the reciprocal function.

These ideas will now be extended to the graphs of quadratic functions and their reciprocals. There are three cases that need to be considered and explored with students, namely, quadratic functions that have one, two or no x-intercepts.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Provide the following table of statements. Ask students to describe the reasoning they used to decide whether each statement is true or false.

Consider the function $f(x)=\dfrac{1}{3x-4}$	True	False	Why I (we) think so
(1) There is a vertical asymptote at $x=\frac{4}{3}$			
(2) There is a horizontal asymptote at $y = 0$.			
(3) $f(0)=\frac{1}{4}$			
(4) The point (4, 8) is on the graph of $y = 3x -4$ and the point (4, -8) is on the graph of $y=\frac{1}{3x-4}$.			

(RF11.1, RF11.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11
7.4 Reciprocal Functions

SB: pp. 392-409
TR: pp. 268-276
BLM: 7-3, 7-7

Relations and Functions

Outcomes

Students will be expected to

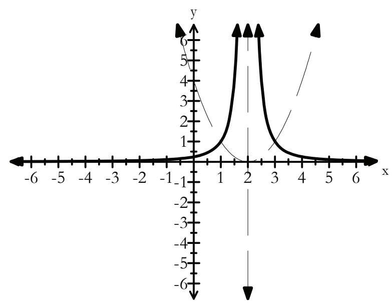
RF11 Continued...
Achievement Indicators:

RF11.1, RF11.2

RF11.3 *Continued*

Elaborations—Strategies for Learning and Teaching

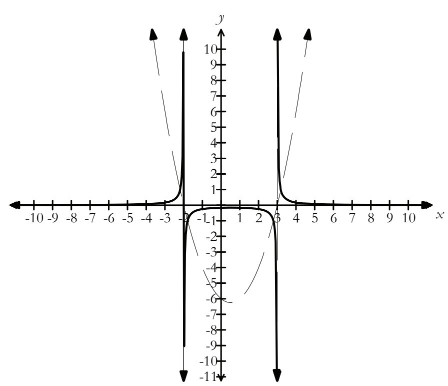
First investigate a quadratic function with one x-intercept. Consider the graph of $y = (x - 2)^2$ and its reciprocal $y = \frac{1}{(x-2)^2}$.



Using the graph as a visual aid, students should observe the following:

- There is one x-intercept on the graph of $y = f(x)$ located at $x = 2$. This corresponds to the location of the vertical asymptote on the graph of $y = \frac{1}{f(x)}$.
- The graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect where $f(x) = \pm 1$(in this case only at $f(x) = 1$). These are the invariant points.
- The graph of $y = \frac{1}{f(x)}$ is also asymptotic to the x-axis. The horizontal asymptote is located at $y = 0$.

Similarly, students can investigate a quadratic function with two x-intercepts. Consider the graph of $y = (x - 3)(x + 2)$ and it's reciprocal $y = \frac{1}{(x-3)(x+2)}$.



Students should note the following features:

- The x-intercepts of the graph of $y = f(x)$, $x = -2$ and $x = 3$ correspond to the location of the vertical asymptotes on the graph of $y = \frac{1}{f(x)}$.
- The graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect at $f(x) = \pm 1$. These are the invariant points.
- The equation of the horizontal asymptote is $y = 0$.
- The parts of the graph of $y = f(x)$ that were positive remain positive on the graph of $y = \frac{1}{f(x)}$. The parts of the graph of $y = f(x)$ that were negative remain negative on the graph of $y = \frac{1}{f(x)}$.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Observation

- Students can work in pairs for this activity. Give one student the graph of a quadratic function and its reciprocal on the same set of axes. Ask him/her to turn to their partner and describe, using the characteristics of the quadratic function and its reciprocal, the graph they see. The other student will draw the graph based on the description from the student. Both students will then check to see if their graphs match.

(RF11.1, RF11.2, RF11.3)

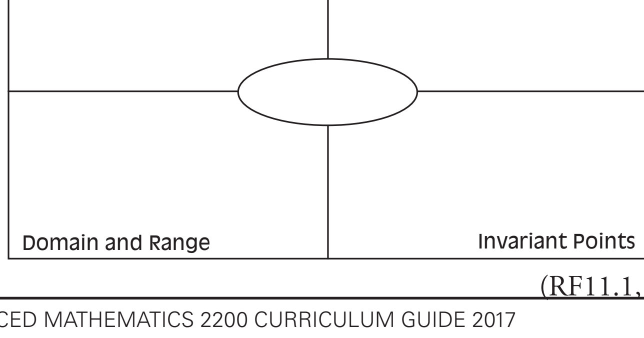
Paper and Pencil

- Provide the following table set of statements. Ask students to describe the reasoning they used to decide whether each statement is true or false.

Consider the function $f(x)=\dfrac{1}{x^2-5x+6}$	True	False	Why I (we) think so
(1) There is a vertical asymptote at x = 2			
(2) There is a vertical asymptote at x = -3			
(3) $f(0)=\frac{1}{6}$			
(4) $y=\frac{1}{f(x)}$ has x-intercepts at 2 and 3			

(RF11.1, RF11.2)

- Ask students to develop Frayer models for given reciprocal functions.



(RF11.1, RF11.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11

7.4 Reciprocal Functions

SB: pp. 392-409
TR: pp. 268-276
BLM: 7-3, 7-7

Relations and Functions

Outcomes

Students will be expected to

RF11 Continued...

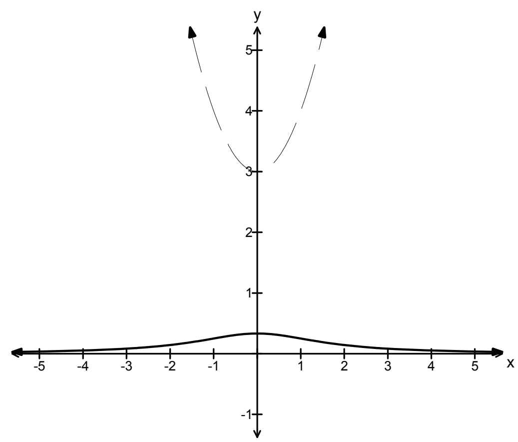
Achievement Indicators:

RF11.1, RF11.2 , RF11.3 *Continued*

RF11.4 Graph, with or without technology, y= f(x) given $y=\frac{1}{f(x)}$ *as a function or a graph, and explain the strategies used.*

Elaborations—Strategies for Learning and Teaching

When graphing a quadratic function with no x-intercepts and its reciprocal, students should notice that the characteristics are different from the previous cases. Consider the graph of $y = x^2 + 3$ and $y=\frac{1}{(x^2+3)}$.



- The graph of $y = f(x)$ has no x-intercepts. Therefore, the graph of $y=\frac{1}{f(x)}$ has no vertical asymptotes.
- The graph of $y = f(x)$ does not have any points at $f(x) = \pm 1$. Therefore, there are no points of intersection between the graphs (i.e., there are no invariant points).
- The equation of the horizontal asymptote is located at $y = 0$.

This investigation helps students recognize the shape of the graph of a reciprocal function. Using the intercepts and asymptotes, students can proceed to graph a reciprocal function.

Students will work backwards and sketch the graph of the original function given the graph of the reciprocal function. They should be able to determine by inspection whether the original function is linear or quadratic by analyzing the reciprocal function. It is important to remind students of the key concepts when working with reciprocal functions. The vertical asymptote of the reciprocal function is the x-intercept(s) of the original graph. The point (x,y) on the reciprocal function becomes $(x, \frac{1}{y})$ on the original function. The two functions will intersect when $f(x) = \pm 1$.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Provide students with a reciprocal function. Ask them to sketch the graph of the original function, $y = f(x)$, and explain the strategies they used.

(RF11.4)

- Ask students:

  How does the graph of $y = \frac{1}{x^2 + a}$ differ when $a > 1$, $a = 1$, and $0 < a < 1$?

(RF11.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11
7.4 Reciprocal Functions

SB: pp. 392-409
TR: pp. 268-276
BLM: 7-3, 7-7

Systems of Equations

Suggested Time: 9 Hours

Unit Overview

Focus and Context

In this unit, students are introduced to linear-quadratic and quadratic-quadratic systems of equations. The solutions to these systems are determined graphically by finding points of intersection. Students will solve the system of equations algebraically using the methods of substitution and elimination. They will then solve application problems involving a system of equations and verify their solutions.

Outcomes Framework

GCO

Develop algebraic and graphical reasoning through the study of relations.

SCO RF6

Solve, algebraically and graphically, problems that involve systems of linear-quadratic and quadratic-quadratic equations in two variables.

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Relations and Functions	Relations and Functions	
RF9 Solve problems that involve systems of linear equations in two variables, graphically and algebraically. [CN, PS, R, T, V]	RF6 Solve, algebraically and graphically, problems that involve systems of linear-quadratic and quadratic-quadratic equations in two variables. [CN, PS, R, T, V]	

Mathematical Processes

[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

Relations and Functions

Outcomes

Students will be expected to

RF6 Solve, algebraically and graphically, problems that involve systems of linear-quadratic and quadratic-quadratic equations in two variables.

[CN, PS, R, T, V]

Achievement Indicators:

RF6.1 *Explain the meaning of the points of intersection of a system of linear-quadratic or quadratic-quadratic equations.*

RF6.2 *Explain, using examples, why a system of linear-quadratic or quadratic-quadratic equations may have zero, one, two or an infinite number of solutions.*

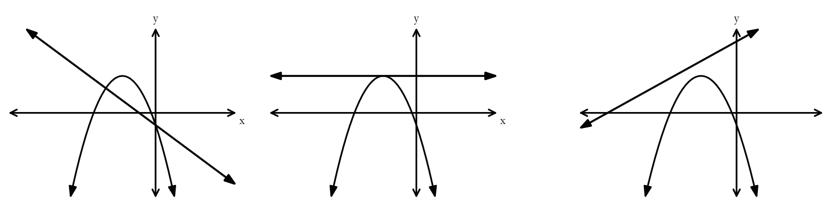
Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students modelled and solved systems of linear equations in two variables, graphically and algebraically (RF9). They created a linear system to model a situation, as well as wrote a description of a situation that might be modeled by a given linear system. They solved linear systems graphically, with and without technology, and progressed to solving linear systems symbolically using substitution and elimination.

In this unit, students' work will be extended to include systems of linear-quadratic and quadratic-quadratic equations.

In Mathematics 1201, students created graphs of linear functions (RF6). They also investigated the meaning of the point of intersection of a system of linear equations (RF9). Students discovered that the intersection points of the graphs represented the solution to the system of equations.

Ask students to sketch a line and a parabola on the same axis. As they examine their intersection points, students should recognize there are three situations possible for a linear-quadratic system. It can have zero, one or two solutions.



Introducing the concepts of a tangent and a secant line may be warranted here. If a line intersects the parabola more than once, the line is referred to as a secant line. If the line intersects the parabola at exactly one point, the line is called a tangent line. Although this is not an outcome in this course, exposure to the terms will help them with courses to follow.

Ask students to repeat the same activity by drawing two parabolas. Ask them how many solutions are possible for a quadratic-quadratic system. Students should recognize the same results with one exception; the system can also have an infinite number of solutions (i.e., coincident).

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to answer the following questions:

  Consider the parabola C whose equation is $y = x^2 - 2x - 3$.

  (i) What is a possible equation for line L such that C and L form a system with no solution? one solution? two solutions?
  (ii) What is a possible equation for quadratic Q such that C and Q form a system with no solution? One solution? Two solutions? Infinite number of solutions?

(RF6.1, RF6.2)

- Ask students how they can tell by observation which of the following systems has no solution and which one has an infinite number of solutions.

  (i) $\left.\begin{aligned}-2x^2+3x-y+4=0\\-4x^2+6x-2y+8=0\end{aligned}\right\}$

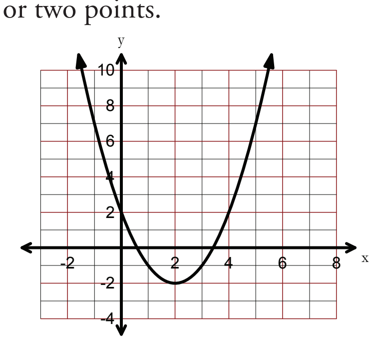
  (ii) $\left.\begin{aligned}y-5=0\\y=-(x+1)^2-3\end{aligned}\right\}$

(RF6.1, RF6.2)

Performance

- Ask students if the following lines intersect the parabola shown below at zero, one or two points.

  (i) $y=\frac{1}{2}x+3$
  (ii) $y=\frac{1}{2}x-4$
  (iii) $y = -2x$
  (iv) $x - 3 = 0$



(RF6.1, RF6.2)

Journal

- Your friend stated that a system of linear-quadratic equations cannot have an infinite number of solutions whereas a system of quadratic-quadratic equations can. Ask students if they agree or disagree with this statement. They should justify their reasoning.

(RF6.1, RF6.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11

8.1 Solving Systems of Equations Graphically

Student Book (SB): pp. 424-439
Teacher Resource (TR): pp. 291-299

Blackline Master (BLM): 8-3, 8-4

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/sys-eqns.html

- graphing software

Relations and Functions

Outcomes

Students will be expected to

RF6 Continued ...

Achievement Indicators:

RF6.3 *Determine and verify the solution(s) of a system of linear-quadratic or quadratic-quadratic equations graphically, with and without technology.*

RF6.4 *Determine and verify the real solution(s) of a system of linear-quadratic or quadratic-quadratic equations algebraically.*

Elaborations—Strategies for Learning and Teaching

Students will be working with the graphs of linear-quadratic and quadratic-quadratic systems to find their solutions graphically. Students will need to revisit the methods used for graphing linear and quadratic relations. In Mathematics 1201, students were exposed to graphing linear functions using the slope-intercept method, slope-point method, and using the x and y-intercept method. In this course, students explored the characteristics of standard form and vertex form of a quadratic function. Identifying the form of the equation will help students decide which method they should choose when graphing the linear or quadratic function. Graphing technology can also be used to solve a system of equations. Students can be exposed to, but not limited to a graphing calculator or graphing apps.

When students identify the points of intersection, remind them to verify the solution for both equations. There are limitations to solving a system by graphing. Non-integral intersection points are possible where students will have to estimate the coordinates. In such cases, an algebraic method of solving these systems is more efficient.

Through their work with solving systems graphically, students should realize that algebra provides a more efficient means of finding the points of intersection. In Mathematics 1201, students were exposed to algebraic methods of substitution and elimination to solve a linear system. They will apply these methods to systems involving quadratic equations, including ones with rational coefficients. Provide students an opportunity to decide which algebraic method is more efficient when solving a system by focusing on the coefficients of like variables. If necessary, ask students to rearrange the equations so that like variables appear in the same position in both equations.

When students solve the system using substitution or elimination, they will have to use factoring or the quadratic formula to solve the resulting quadratic equation. After determining the solution to the system, ensure that students verify the ordered pair satisfies the equations. Discuss with them why it is important to verify the solution in both equations.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Kendra solved the system: $\left.\begin{array}{l} 2x - y = 9 \\ y = x^2 - 4x \end{array}\right\}$

  Her solution was (3,-3). Ask students to verify whether her solution is correct. Ask them to explain how Kendra's results can be illustrated on a graph.

  (RF6.3)

- Andrew was asked to solve the system: $\begin{cases} x^2 - x + y = -2 \\ 2x^2 - 4x + 3y = 0 \end{cases}$

  The beginning of his solution is shown below. Ask students to finish the solution and verify their answer. Identify and correct any error(s) you or Andrew may have made.

  Andrew's Solution:

  Multiply the first equation by -3 and add the second equation.

  $$x^2 - x + y = -2 \xrightarrow{\times(-3)} -3x^2 + 3x - 3y = -2$$
  $$2x^2 - 4x + 3y = 0 \longrightarrow \underline{\;2x^2 - 4x + 3y = 0\;}$$
  $$-x^2 - x \qquad = -2$$

  Now, solve $-x^2 - x = -2$.

  (RF6.4)

Observation

- Ask students to participate in the following activity: Cards are distributed amongst the students. One third of the cards contains the coordinates of the points of intersection of two graphs. Another third contains a system of linear-quadratic or quadratic-quadratic equations written in slope *y*-intercept and vertex form. The final third contains the corresponding graphs for the system of equations. Students move around the classroom attempting to form a group of three by matching the cards containing the corresponding systems, graphs and points of intersections.

  (RF6.3, RF6.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11

8.1 Solving Systems of Equations Graphically

SB: pp. 424-439
TR: pp. 291-299
BLM: 8-3, 8-4

8.2 Solving Systems of Equations Algebraically

SB: pp. 440-456
TR: pp. 300-306
BLM: 8-3, 8-5

Relations and Functions

Outcomes

Students will be expected to

RF6 Continued ...

Achievement Indicators:

RF6.4 *Continued*

RF6.5 *Model a situation, using a system of linear-quadratic or quadratic-quadratic equations.*

RF6.6 *Relate a system of linear-quadratic or quadratic-quadratic equations to the context of a given problem.*

Elaborations—Strategies for Learning and Teaching

Encourage students to predict the number of solutions before they solve the system graphically or algebraically. The discriminant, $b^2 - 4ac$, can be used to help them with their prediction. Consider the system: $y = 3x + 5$ and $y = 3x^2 - 2x - 4$. Ask students to equate the equations and simplify. Their resulting quadratic equation is $3x^2 - 5x - 9 = 0$. They determine the value of the discriminant to be 133. Students should realize that when the discriminant is greater than zero, there are two x values indicating two different solutions. This means that the linear-quadratic system has two points of intersection. If the discriminant is equal to zero, there is one solution and a negative discriminant implies no point of intersection. For systems with real solutions, students would then proceed to solve the system and the resulting quadratic equation using methods learned earlier in the Quadratic Equations unit (factoring, quadratic formula, completing the square). Although students were briefly exposed to complex numbers in the quadratic unit, students are not required to find the non-real solutions to a system of equations.

Remind students that a system of quadratic-quadratic equations can also produce an infinite number of solutions. When the functions are coincident (i.e., identical) students will recognize this results in a system where the left hand side equals the right hand side (i.e., $0 = 0$).

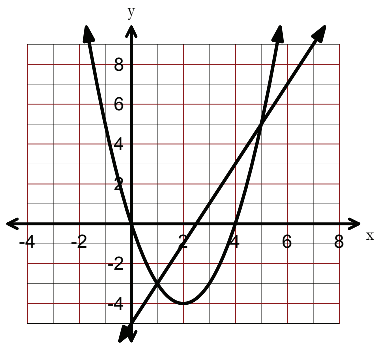
Students will model situations using a system of linear-quadratic and quadratic-quadratic equations connected to a variety of real-life contexts. When they can relate their learning to real-life applications it has more meaning. Paths of thrown or falling objects can be used to model both quadratic and linear functions. Ensure students understand and define the variables that are being used to represent the unknown quantities. Discuss with students that in order to solve a system, the number of unknowns must match the number of equations.

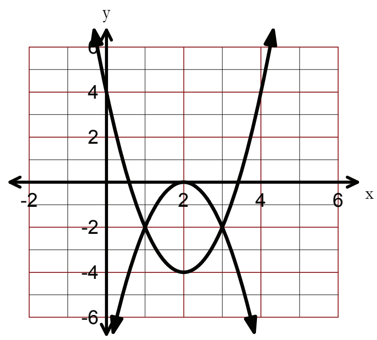
General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

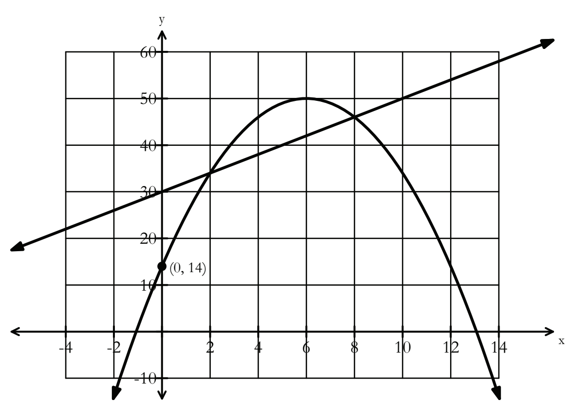
- Ask students to create and solve the system of equations represented by the graphs below. They should verify that the solutions obtained algebraically match those found graphically.





(RF6.3, RF6.4)

- The price C, in dollars per share, of a high-tech stock has fluctuated over a twelve-year period and is represented by the parabola shown. The price C, in dollars per share, of a second high-tech stock has shown a steady increase during the same time period.



Ask students to answer the following:

(i) Determine the system of equations that models the price over time.
(ii) Solve the system.
(iii) Determine the values where the two prices are the same.

(RF6.5, RF6.6)

Resources/Notes

Authorized Resource

Pre-Calculus 11

8.2 Solving Systems of Equations Algebraically

SB: pp. 440-456
TR: pp. 300-306
BLM: 8-3, 8-5

Relations and Functions

Outcomes

Students will be expected to

RF6 Continued...

Achievement Indicator:

RF6.7 *Solve a problem that involves a system of linear-quadratic or quadratic-quadratic equations, and explain the strategy used.*

Elaborations—Strategies for Learning and Teaching

Students will solve problems involving a system of linear-quadratic or quadratic-quadratic equations. Students should decide which algebraic method is more efficient when solving the system of equations and they should be able to explain why they chose the preferred method.

Explaining the meaning of a solution in a particular context and verifying the solution is an important part of solving systems. An answer to a system of equations may not necessarily be a possible solution in the context of the situation (i.e., inadmissible root). While solving a system of equations, a student may determine that the solutions are, for example, (-2, 5) and (5, 8). In the context of the problem the *x*-coordinate may represent a measurement such as time or length. The point (-2, 5), therefore, would have to be rejected as a solution leading to only one possible answer.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to solve the following:

  (i) A rectangular field has a perimeter of 500 m and an area of 14 400 m^2. Find the length of the sides.

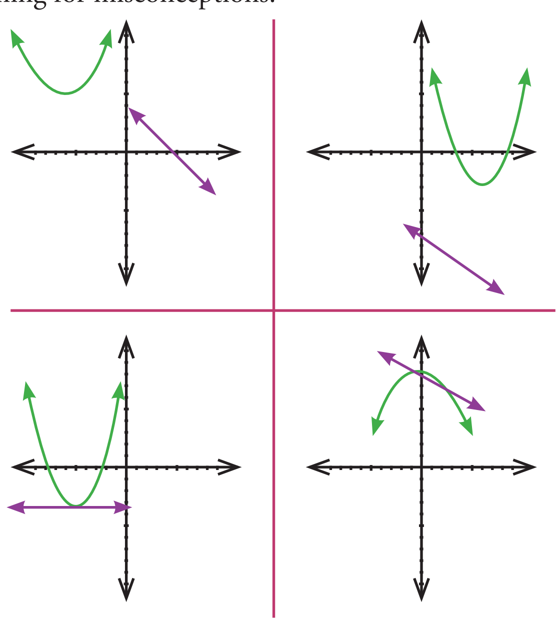
  (ii) Ask students to write a system of equations to represent two numbers that differ by 4 and whose squares have a sum of 136?

  (iii) The revenue for a production by your school drama group is $R = -50t^2 + 300t$, where t is the ticket price in dollars. The cost for the production is $C = 600 - 50t$. Determine the minimum ticket price that will allow the production to break even (i.e., a production breaks even when revenue = cost)

(RF6.7)

Observation

- Invite students to play WODB (Which One Doesn't Belong?). Show students the following graphs. Each graph could be the one which doesn't belong, but for a different reason. Observe students' reasoning for misconceptions.



(RF6.2, RF6.7)

Resources/Notes

Authorized Resource

Pre-Calculus 11

8.2 Solving Systems of Equations Algebraically

SB: pp. 440-456
TR: pp. 300-306
BLM: 8-3, 8-5

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/sys-eqns.html

- WODB Graphs

Linear and Quadratic Inequalities

Suggested Time: 9 Hours

Unit Overview

Focus and Context

In this unit, students will solve quadratic inequalities in one variable as well as linear and quadratic inequalities in two variables. They will investigate finding the solutions using various methods, ideally leading to a preferred personal strategy. They will apply the various skills developed to solve a number of real-world problems.

Outcomes Framework

GCO

Develop algebraic and graphical reasoning through the study of relations.

SCO RF7

Solve problems that involve linear and quadratic inequalities in two variables.

SCO RF8

Solve problems that involve quadratic inequalities in one variable.

SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Relations and Functions	Relations and Functions	Relations and Functions
RF6 Relate linear equations expressed in: • slope-intercept form ($y = mx + b$) • general form ($Ax + By + C = 0$) • slope-point form ($y - y_1 = m(x - x_1)$) to their graphs. [CN, R, T, V]	RF7 Solve problems that involve linear and quadratic inequalities in two variables. [C, PS, T, V] RF8 Solve problems that involve quadratic inequalities in one variable. [CN, PS, V]	RF11 Graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5). [C, CN, T, V]

Mathematical Processes

[C]	Communication	**[PS]**	Problem Solving
[CN]	Connections	**[R]**	Reasoning
[ME]	Mental Mathematics and Estimation	**[T]**	Technology
		[V]	Visualization

Relations and Functions

Outcomes

Students will be expected to

RF7 Solve problems that involve linear and quadratic inequalities in two variables.

[C, PS, T, V]

Achievement Indicators:

RF7.1 *Explain, using examples, how test points can be used to determine the solution region that satisfies a linear inequality.*

RF7.2 *Explain, using examples, when a solid or broken line should be used in the solution for a linear inequality.*

RF7.3 *Sketch, with or without technology, the graph of a linear inequality.*

Elaborations—Strategies for Learning and Teaching

In Grade 9, students solved single variable linear inequalities with rational coefficients and graphed their solutions on a number line (9PR4). Students are familiar with the inequality symbols, the terms continuous and discrete data, as well as their effect on a graph. In Mathematics 1201, students graphed linear relations expressed in slope-intercept form, general form and point-slope form (RF6).

Students are familiar with verifying solutions to linear equations and linear inequalities using substitution. They also verified solutions by examining the number line graphs of inequalities. Students have been exposed to the concept that the solution of a linear inequality consists of a set of points while the solution of a linear equation has only one solution.

In this unit, students will solve quadratic inequalities in one variable, as well as linear and quadratic inequalities in two variables.

Students will first investigate linear inequalities with two variables. The solution of a linear inequality in two variables, such as $Ax + By > C$, consists of any ordered pair (x, y) that produces a true statement when the values of x and y are substituted into the inequality. Since there may be infinitely many solutions, it is not possible for students to list all of them. Discussion around the difference between an equation and an inequality should highlight that the solution to an inequality in two variables is represented by a region. Explain to students that the boundary line $Ax + By = C$ divides the coordinate plane into two parts, where one part represents the solutions to the inequality.

To be successful with graphing inequalities in two variables, students must be proficient with graphing linear equations, with and without technology. Identifying the form of the equation will help students decide which method they should choose when graphing the line (i.e., slope-intercept, slope-point or x-and y-intercept method). Initially, students use test points to investigate the region of the plane that satisfies the inequality which will help guide them to an understanding of when to shade above or below the boundary. Working through several examples should help them make the connection between the sign of the inequality, the shaded region and whether to use a solid or broken line.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to complete a table similar to the one below:

Inequality	Shade above or below?	Broken or solid line?
$y \leq -2x + 8$		
$x + y > 2$		
$4x - 2y \geq 12$		

(RF7.2, RF7.3)

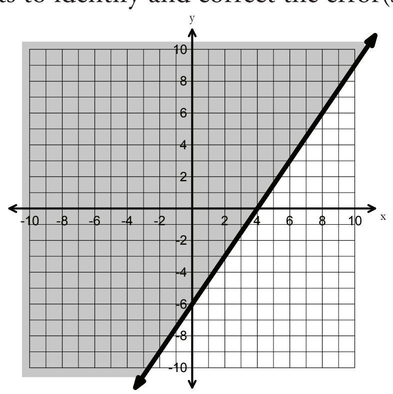
- A student was asked to graph $3x - 2y > 12$. The solution is shown below. Ask students to identify and correct the error(s).

Student Solution:

$3x - 2y > 12$

$-2y > -3x + 12$

$y > \frac{3}{2}x - 6$



(RF7.3)

Performance

- Ask students to work in pairs for this activity. Each pair should be given a deck of cards consisting of 12 cards with inequalities and 12 cards with the corresponding graphs. Students make pairs consisting of a graph and its matching inequality. These pairs are set aside face up on the table. Students take turns selecting a card from their partner and making matches if possible. Play continues until all matches are made.

(RF7.3)

Journal

- Your friend asks you to explain the difference between graphing $3y + 2x = 4$, $3y + 2x > 4$, and $3y + 2x \geq 4$. Ask students to write a response.

(RF7.2)

- The solution region for the inequality $5x - 3y > 10$ is above the line since it contains a "greater than sign". Ask students if they agree or disagree with their statement. They should explain their reasoning.

(RF7.3)

Resources/Notes

Authorized Resource

Pre-Calculus 11

9.1 Linear Inequalities in Two Variables

Student Book (SB): pp. 464-475
Teacher Resource (TR): pp. 317-323
Blackline Master (BLM): 9-3, 9-4

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/ineqls.html

- inequalities card game

Relations and Functions

Outcomes

Students will be expected to

RF7 Continued ...

Achievement Indicators:

RF7.1, RF7.2, RF7.3
Continued

RF7.4 *Solve a problem that involves a linear inequality.*

Elaborations—Strategies for Learning and Teaching

When solving inequalities, students sometimes forget to reverse the sign of the inequality when multiplying or dividing both sides by a negative number. A brief review may be necessary as this error will also affect the feasible region when solving inequalities in two variables.

Students should be provided with opportunities to write the equation of the linear inequality given its graph. They will find the equation of the boundary line and then use the given shaded region to determine the correct inequality. Remind students what a broken or solid boundary line represents.

Students will relate inequalities to real life contexts. Problems that can be expressed as an inequality in two variables require students to find two unknown quantities under certain constraints. Students will translate the word problem into an inequality. Ensure that they understand and define the variables that are being used to represent the unknown quantities. They should also realize that the shaded feasible region represents all possible combinations for the two quantities. An example such as the following could be used to highlight feasible and realistic solutions to a problem.

- With two minutes left in a basketball game, your team is 12 points behind. What are two different numbers of 2-point and 3-point shots your team could score to earn at least 12 points?

Students should recognize that the domain and range for this problem contain only positive values. Ask them to identify realistic points in the feasible region considering the time remaining in the game.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to answer the following:
  - (i) Susan plans to spend a maximum of 15 hours reviewing Math and Biology in preparation for examinations. Draw a graph showing how much time she could spend studying each subject.

    (RF7.4)
  - (ii) A contractor has at least one hundred tonnes of soil to be moved using two trucks. One truck has a 4 ton capacity and the other has a 5 ton capacity. Make a graph to show the various combinations of loads the two trucks could carry to complete the job.

    (RF7.4)

Observation

- Invite students to play WODB (Which One Doesn't Belong?). Show students the following inequalities. Each inequality could be the one which doesn't belong, but for a different reason. Observe students' reasoning for misconceptions.

$2x+3y<-6$	$2x+3y\geq -12$
$2x-3y\geq 3$	$2x+3y\leq 6$

(RF7.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11

9.1 Linear Inequalities in Two Variables

SB: pp. 464-475
TR: pp. 317-323
BLM: 9-3, 9-4

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/ineqls.html

- WODB inequalities

Relations and Functions

Outcomes

Students will be expected to

RF8 Solve problems that involve quadratic inequalities in one variable.
[CN, PS, V]

Achievement Indicators:

RF8.1 Determine the solution of a quadratic inequality in one variable, using strategies such as case analysis, graphing, roots and test points, or sign analysis; and explain the strategy used.

RF8.2 *Represent and solve a problem that involves a quadratic inequality in one variable.*

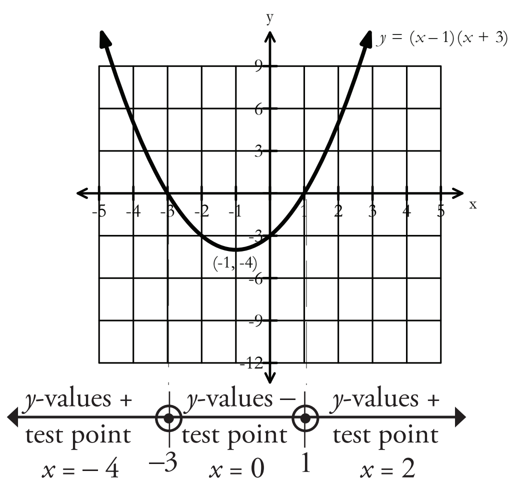
RF8.3 *Interpret the solution to a problem that involves a quadratic inequality in one variable.*

Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students were introduced to domain and range (RF1). They used interval notation and set notation to represent solution sets. This will now be extended to solving quadratic inequalities.

It is important to have a discussion with students about what it means to solve a quadratic inequality in one variable. When solving $ax^2 + bx + c \geq 0$, for example, students should interpret this as finding the possible x-values where the corresponding y-values are zero or positive. In other words, when is the graph of $y = ax^2 + bx + c$ on or above the x-axis. This can be done graphically or algebraically. Students may have a better understanding of the algebraic method, however, if it is related back to the graph. Regardless of the technique used, proficiency in solving quadratic equations is important.

One strategy that can be used to solve a quadratic inequality involves roots and test points. Solving an inequality such as $x^2 + 2x - 3 < 0$, students first determine the roots of the quadratic equation $x^2 + 2x - 3 = 0$. They will then use a sign diagram consisting of a number line and test points to determine the intervals that satisfy the inequality. The sign diagram was introduced in the Absolute Value and Reciprocal Unit.



Relating the sign diagram to the x-axis of the graph, students use test points to determine if the function is positive or negative. They should recognize the roots -3 and 1 are not part of the solution. Students can write the solution set as set notation, $\{x \mid -3 < x < 1, x \in \mathbb{R}\}$ or interval notation (-3, 1).

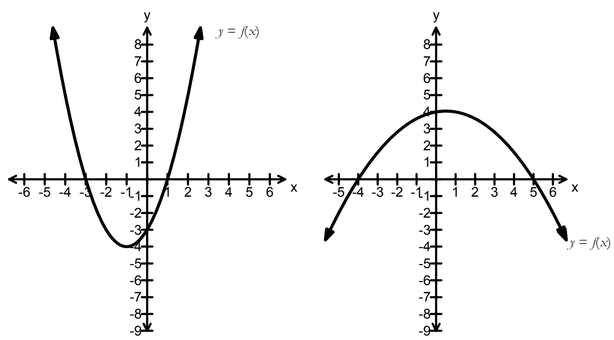
Students will solve problems that involve a quadratic inequality in one variable. These types of problems will be similar to solving a quadratic equation except there is usually a minimum or maximum constraint. Students should relate their solution to the context of the problem.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to answer the following:
  - (i) Given A = x^2 - 7 and B = -4x + 5, for what values of x is A < B?
  - (ii) Use the graphs shown below to state the x-values for which $f(x) > 0$.



  - (iii) When a projectile is fired into the air, its height h, in metres, t seconds later is given by the equation h = 11t - 3t^2. When is the projectile at least 6 m above the ground?
  - (iv) When a baseball is hit by a batter, the height of the ball, $h(t)$, in feet after t seconds, is determined by the equation $h(t) = -16t^2 + 64t + 4$. For which interval of time is the height of the ball greater than or equal to 52 feet?
  - (v) A rectangular solid has a length 3 cm more than the width. The height is 4 cm. What possible dimensions would result in a surface area that is less than 144 cm^2?

(RF8.1, RF8.2, RF8.3)

Journal

- Your friend asks you to explain the difference between solving $x^2 - 2x - 3 = 0$, $x^2 - 2x - 3 \geq 0$, and $x^2 - 2x - 3 < 0$. Ask students to write a response.

(RF8.1, RF8.2)

Resources/Notes

Authorized Resource

Pre-Calculus 11

9.2 Quadratic Inequalities in One Variable

SB: pp. 476-487
TR: pp. 324-329
BLM: 9-3, 9-5

Relations and Functions

Outcomes

Students will be expected to

RF7 Continued...

Achievement Indicators:

RF7.5 *Explain, using examples, how test points can be used to determine the solution region that satisfies a quadratic inequality.*

RF7.6 *Explain, using examples, when a solid or broken line should be used in the solution for a quadratic inequality.*

RF7.7 *Sketch, with or without technology, the graph of a quadratic inequality.*

RF7.8 *Solve a problem that involves a quadratic inequality.*

Elaborations—Strategies for Learning and Teaching

The strategies used to graph linear inequalities in two variables will now be extended to quadratic inequalities in two variables. Solving quadratic inequalities in one variable involved students finding the set of all x values that made a particular inequality true. Solving a quadratic inequality in two variables implies finding, graphically, all the coordinate pairs (x, y) that make a particular inequality true.

The graph of a quadratic function separates the plane into two regions, one of which contains all the points that satisfy the inequality. Students will graph the corresponding function that is associated with the inequality and use test points to see which region should be shaded. They should continue to ask themselves if the graph should be drawn with a solid or broken boundary.

Students will also write an inequality to describe a graph, given the function defining its boundary. Ensure students recognize whether the parabola is solid or broken. Ask them how to identify the inequality using test points.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to sketch the feasible region for each of the following inequalities.

  (i) $y \geq x^2 + 2x - 3$
  (ii) $y < -2(x-1)^2 + 4$
  (iii) $y < \frac{1}{3}(x-3)(x+3)$

  (RF7.7)

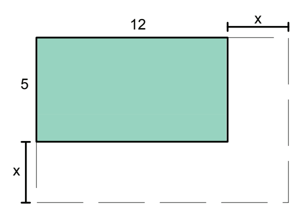
- Give students a number of linear and quadratic inequalities such as:

  (i) $2x - 3y < 7$
  (ii) $-x + 4y \geq -6$
  (iii) $y > -2(x + 1)^2 - 4$
  (iv) $y \leq -x^2 + 1$

  Students can work in groups. Provide them with a list of 5 points and have them work together to determine which inequality, if any, satisfies their point.

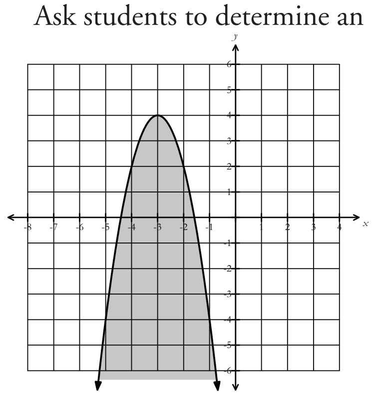
  (RF7.1, RF7.5)

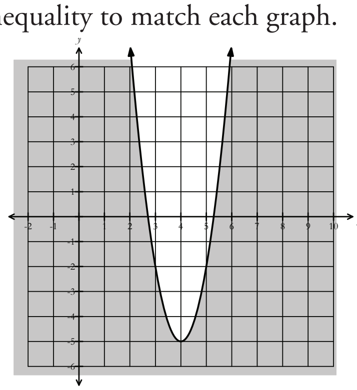
- The base of a rectangular bin currently has dimensions 12 m by 5 m. The base is to be enlarged by an equal amount on the width and length so that the area is more than doubled. Ask students by how much should the length and width be increased to produce the desired area.



(RF7.8)

- Ask students to determine an inequality to match each graph.





(RF7.6)

Resources/Notes

Authorized Resource

Pre-Calculus 11

9.3 Quadratic Inequalities in Two Variables

SB: pp. 488-500
TR: pp. 330-336
BLM: 9-3, 9-6

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/ineqls.html
- graphs of inequalities

Sequences and Series

Suggested Time: 11 Hours

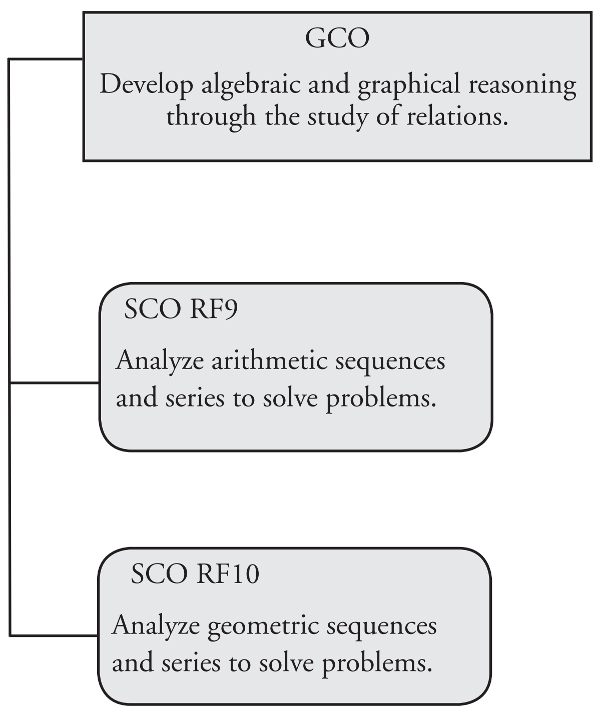
Unit Overview

Focus and Context

In this unit students will differentiate between a finite and infinite sequence. They will then explore and develop a formula for the general term for arithmetic and geometric sequences and progress to arithmetic and geometric series.

Students will determine if a given series is convergent or divergent. They will then write the general equation for the sum of an infinite geometric series. The concepts of sequences and series will be applied to solve problems.

Outcomes Framework



SCO Continuum

Mathematics 1201	Mathematics 2200	Mathematics 3200
Relations and Functions	Relations and Functions	Relations and Functions
	RF9 Analyze arithmetic sequences and series to solve problems. [CN, PS, R] RF10 Analyze geometric sequences and series to solve problems. [PS, R]	

Mathematical Processes

[**C**]	Communication	[**PS**]	Problem Solving
[**CN**]	Connections	[**R**]	Reasoning
[**ME**]	Mental Mathematics and Estimation	[**T**]	Technology
		[**V**]	Visualization

Relations and Functions

Outcomes

Students will be expected to

RF9 Analyze arithmetic sequences and series to solve problems.

[CN, PS, R]

Achievement Indicators:

RF9.1 *Identify the assumption(s) made when defining an arithmetic sequence or series.*

RF9.2 *Provide and justify an example of an arithmetic sequence.*

RF9.3 *Determine a rule for finding the general term of an arithmetic sequence.*

Elaborations—Strategies for Learning and Teaching

In Grade 9, students were exposed to linear patterns (9PR1). In Mathematics 1201, they determined the equation of a linear relation (RF7). In this unit, students will explore the concept of arithmetic sequences and series. They will work with patterns where consecutive terms produce a common difference. Students will write a formula for the general term of an arithmetic sequence and solve for any missing values.

Once sequences have been explored, students will be introduced to an arithmetic series. They will make the connection that an arithmetic series is the sum of an arithmetic sequence, derive a formula and apply it in a variety of problem solving situations.

Students should investigate the concept of a sequence by observing a variety of number patterns to develop an understanding of the notation, symbols and domain associated with arithmetic sequences (i.e. terms t_1, t_2, finite sequence, infinite sequence, common difference d, and the general term t_n).

In Mathematics 1201 students were exposed to the concept of domain as it relates to linear functions (RF1). This may be a good opportunity to provide students with an arithmetic sequence and ask them to graph the terms of the sequence. Students should make the connection that the terms of the sequence are the ordered pairs of the graph. Ask them what the x and y-values of the ordered pair represent and what number system they belong to. They should recognize the domain of the sequence is the set of natural numbers.

Encourage students to provide their own examples of arithmetic sequences and justify, finding the difference between consecutive terms, why such a sequence is arithmetic. They should not assume every sequence that has an increasing or decreasing pattern in addition is arithmetic. Students should conclude, however, that if a sequence is arithmetic you are always adding the same number each time.

Use a numerical example to give students an opportunity to identify patterns and a possible rule for an arithmetic sequence. Consider a sequence such as 5, 7, 9, 11, 13,...... and the following questions to guide student discussion.

- What is the common difference?
- Can this sequence be rewritten to show the pattern of the first term and the common difference?
- Can you predict the formula for the general term of an arithmetic sequence based on the pattern 5, 5 + 2, 5 + 2(2), 5 + 3(2),.....?
- Can you write the pattern in general terms for any first term and common difference ?

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to determine the first five terms for each of the sequences.

  (i) $t_n = 2n - 1$

  (ii) $t_n = \sqrt{4n^2 - 4n + 1}$

  (iii) $t_n = \frac{2n^2+n-1}{n+1}$

  Ask them if each sequence is arithmetic since the first five terms result in {1, 3, 5, 7, 9}. They should explain their reasoning.

(RF9.2)

Observation

- Ask students to explore the following sequences to determine whether each is arithmetic. They should justify their conclusions.

  (i) 5.3, 5.9, 6.5
  (ii) $x - 1, x + 1, x + 3$
  (iii) x_1, x_2, x_3
  (iv) $2x + 5, 4x + 5, 6x + 5$

(RF9.2)

Journal

- Ask students to explain the term “sequence” in their own words. Ask them to compare this to a mathematical definition found online. They should explain how the two definitions are similar and/or different.

(RF9.2)

Resources/Notes

Authorized Resource
Pre-Calculus 11

1.1 Arithmetic Sequences

Student Book (SB): pp. 6-21
Teacher Resource (TR): pp. 10-18
Blackline Master (BLM): 1-3, 1-4

Relations and Functions

Outcomes

Students will be expected to

RF9 Continued ...

Achievement Indicators:

RF9.2, RF9.3 *Continued*

RF9.4 *Determine t_1, d, n, or t_n in a problem that involves an arithmetic sequence.*

RF9.5 *Solve a problem that involves an arithmetic sequence or series.*

RF9.6 *Describe the relationship between arithmetic sequences and linear functions.*

Elaborations—Strategies for Learning and Teaching

Students should notice that the terms of the arithmetic sequence with first term t_1 and a common difference d can be written as $t_1 = t_1$, $t_2 = t_1 + d$, $t_3 = t_1 + 2d$, $t_4 = t_1 + 3d$... Using this pattern, they can write the formula for any arithmetic sequence $t_n = t_1 + (n - 1)d$.

Give students an opportunity to write their own examples of arithmetic sequences and to write the appropriate formula. They should use numerical and algebraic examples. Encourage them to list t_1, d, and n, and then write the equation for t_n.

When solving problems involving arithmetic sequences consider a variety of examples such as, but not limited to, the following:

- Finding the general term t_n using tables, charts, graphs or an equation
- Finding the number of terms in a finite arithmetic sequence when given the value of the n^{th} term
- Finding the common difference when provided with algebraic expressions representing the value of terms for the sequence

When provided with the general term, students sometimes have difficulty differentiating between the term and the term number. When they are asked to find the tenth term, for example, they may be unsure whether to write t_{10} or $t_n = 10$. It is important to reinforce that t_{10} represents the tenth term while $t_n = 10$ represents the n^{th} term having a value of 10.

Students will make the connection that arithmetic sequences, when graphed, form linear graphs. Students may need to review graphing linear functions using a table of values and/or the slope y-intercept method. They should observe the following features when comparing an arithmetic sequence to a linear function:

- The slope is the common difference.
- The y-intercept is the initial value minus the common difference.
- The domain of a sequence is the set of natural numbers, while the domain of a linear function, based on the context, can be the set of real numbers.
- The graph of an arithmetic sequence consists of discrete data, while the graph of a linear function may be discrete or continuous.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to create two arithmetic sequences and find the general term t_n. Ask them to determine the sum and then the product of the two sequences. Explain if these new sequences created are arithmetic and justify your reasoning.

(RF9.2, RF9.4)

- For each arithmetic sequence, ask students to write a formula for t_n and use it to find the indicated term.

(i) -4, 1, 6, 11, … , t_{13}

(ii) 9, 1, -7, -15, …, t_{46}

(RF9.4, RF9.5)

- Consecutive terms of an arithmetic sequence are $(5 + x)$, 8, and $(1 + 2x)$. Ask students to determine the value of x.

(RF9.4, RF9.5)

- Ask students to answer the following:
The diagram shows a pattern of positive integers in five columns. If the pattern is continued, in which columns will these numbers appear?

(i) 49

(ii) 117

(iii) 301

(iv) 8725

Col 1	Col 2	Col 3	Col 4	Col 5
1		2		3
	5		4	
6		7		8
	10		9	
11		12		13
	15		14	

(RF9.4, RF9.5)

Interview

- Ask students to consider the following problem and explain if the calculation is correct. They should justify their reasoning.

Paul must determine the 50th term of an arithmetic sequence beginning with 5 and having a common difference of 9. He calculates (50 x 9) + 5 = 455.

(RF9.4, RF9.5)

- Zachary is having trouble remembering the formula $t_n = t_1 + (n - 1)d$ correctly. He thinks the formula should be $t_n = t_1 + nd$. Ask students how they would explain to Zachary that he should use $(n - 1)d$ rather than nd in the formula.

(RF9.4, RF9.5)

Resources/Notes

Authorized Resource

Pre-Calculus 11

1.1 Arithmetic Sequences

SB: pp. 6-21
TR: pp. 10-18
BLM: 1-3, 1-4

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/seqser.html

- table with number pattern

Relations and Functions

Outcomes

Students will be expected to

RF9 Continued ...

Achievement Indicators:

RF9.6 *Continued*

Elaborations—Strategies for Learning and Teaching

The relationship between an arithmetic sequence and a linear function can be highlighted through a discussion such as the following:

n	1	2	3	4	5
t_n	3	5	7	9	11

- Find the general term t_n. Compare this equation to a linear equation $y = mx + b$. Are they similar?
- Graph the sequence. What do you notice?
- What is the relationship between the slope of the line and the common difference?
- Evaluate the first term minus the common difference of the sequence. What does this value represent in the linear equation?

The relationship between arithmetic sequences and linear functions could also be developed using a symbolic approach.

$$t_n = t_1 + (n - 1)d$$
$$t_n = t_1 + dn - d$$
$$t_n = dn + t_1 - d$$
$$t_n = dn + (t_1 - d) \quad \text{vs} \quad y = mx + b$$
where $d = m$ and $t_1 - d = b$

RF9.7 *Determine a rule for finding the sum of n terms of an arithmetic series.*

Students will write the sum of the first n terms for any arithmetic sequence. They should work with a numerical example and based on their observations, write a formula for the general case. The following example provides an opportunity to review arithmetic sequences and introduce arithmetic series.

- Suppose you create a Facebook™ account and add one new friend on day 1, four more friends on day 2, four more friends on day 3 and so on for a total of 25 days. How many friends would you have in total after the 25th day?

Use the following questions to guide students:

- What is the arithmetic sequence?
- What is the first term?
- What is the rule for t_n? What is the last term?
- What do the sum of the terms in the sequence represent?
- How many terms are we summing up?
- How can the value of S_{25} be determined?

Although some students may just add up all the terms in the sequence, they should realize this is not the most efficient method.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to complete the following activity using technology:

n	1	2	3	4	5
t_n	1	-1	-3	-5	-7

(i) Find a formula for the general term t_n and record your findings.

(ii) Produce a scatterplot for the data.

(iii) Produce and plot the line of best fit and record the equation.

(iv) Compare your results from steps (i) and (iii). How does the common difference relate to the slope?

(v) Subtract d from the first term of the sequence. Compare this value to the y-intercept from the line of best fit. What do you notice?

(RF9.6)

Resources/Notes

Authorized Resource

Pre-Calculus 11

1.1 Arithmetic Sequences

SB: pp. 6-21
TR: pp. 10-18
BLM: 1-3, 1-4

1.2 Arithmetic Series

SB: pp. 22-31
TR: pp. 19-25
BLM: 1-3, 1-5

Relations and Functions

Outcomes

Students will be expected to

RF9 Continued ...

Achievement Indicator:

RF9.7 *Continued*

Elaborations—Strategies for Learning and Teaching

Introduce students to Gauss's method of determining the sum of the first 100 positive integers. Use this method to determine S_{25} in the previous example. Ask students to write the series twice, once in ascending order and the other in descending order. Then, sum the two series.

$$\begin{aligned}
S_{25} &= 1 + 5 + 9 + \ldots\ldots + 89 + 93 + 97 \\
+S_{25} &= 97 + 93 + 89 + \ldots\ldots + 9 + 5 + 1 \\
\hline
2S_{25} &= 98 + 98 + 98 + \ldots\ldots + 98 + 98 + 98 \\
2S_{25} &= 25(98) \\
S_{25} &= \frac{25}{2}(98)
\end{aligned}$$

Encourage students to try to figure out where the different parts of that formula come from. When they are trying to make a conjecture about the general case, it is best to leave the original values rather than simplifying the expression. Ask students what they think the values 25 and 98 represent. They should recognize that 25 is the number of terms and 98 is the sum of 1 and 97, representing t_1 and t_{25}. Therefore, $S_{25} = \frac{25}{2}(t_1 + t_{25})$.

The algebraic approach should also be explored to derive the formula for the sum of the general arithmetic series $S_n = \frac{n}{2}(t_1 + t_n)$. The sum of n terms of an arithmetic series is represented by $S_n = t_1 + t_1 + d + t_1 + 2d + \ldots\ldots\ t_1 + (n - 1)d$. Using Gauss's method, guide students through the following derivation:

$$\begin{aligned}
S_n &= t_1 + t_1 + d + \ldots\ldots\ t_1 + (n-2)d + t_1 + (n-1)d \\
S_n &= t_1 + (n-1)d + t_1 + (n-2)d + \ldots\ldots\ t_1 + d + t_1 \\
2S_n &= 2t_1 + (n-1)d + 2t_1 + (n-1)d + \ldots\ldots\ 2t_1 + (n-1)d + 2t_1 + (n-1)d \\
2S_n &= n(2t_1 + (n-1)d) \\
2S_n &= n(t_1 + \underbrace{t_1 + (n-1)d}_{t_n}) \\
2S_n &= n(t_1 + t_n) \\
S_n &= \frac{n(t_1 + t_n)}{2}
\end{aligned}$$

Students should not be required to reproduce this derivation for assessment purposes.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- Ask students to use Gauss's method to determine the sum of the first 12 positive integers.

(RF9.7)

Resources/Notes

Authorized Resource

Pre-Calculus 11

1.2 Arithmetic Series

SB: pp. 22-31
TR: pp. 19-25
BLM: 1-3, 1-5

Relations and Functions

Outcomes

Students will be expected to

RF9 Continued ...
Achievement Indicators:

RF9.8 *Determine t_1, d, n, or S_n in a problem that involves an arithmetic series.*

RF9.5 *Continued*

Elaborations—Strategies for Learning and Teaching

Students will evaluate and/or manipulate the given arithmetic series to determine the first term, the common difference, the number of terms in the sequence, the n^{th} term or the sum. They should solve problems that involve an arithmetic series within a context. Consider the following example:

- An auditorium has 20 seats on the first row, 24 seats on the second row, 28 seats on the third row, and so on and has 30 rows of seats. How many seats are in the theatre?

To solve this problem, students need to ask themselves the following questions:

- What is the problem asking?
- What is the pattern? Does it represent an arithmetic sequence?
- What information do we need to find?

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Paper and Pencil

- A round robin tournament is to be played by 3 teams. Ask students to determine the total number of games required to guarantee that each team plays the other exactly once. Repeat the same activity for a tournament consisting of 4 teams. Ask students to use their findings from above to find the total number of games for a national round robin tournament with all ten provinces and three territories participating. They should explain how their answer is related to an arithmetic series.

(RF9.5, RF9.8)

- Ask students to find the sum of 20 + 14 + 8 + … + (-70).

(RF9.8)

- A theater has 60 seats in the first row, 68 seats in the second row, 76 seats in the third row, and so on in the same increasing pattern. If the theater has 20 rows of seats, how many seats are in the theater?

(RF9.5, RF9.8)

- In an arithmetic series $t_1 = 6$ and $S_9 = 108$. Ask students to find the common difference and the sum of the first 20 terms.

(RF9.8)

- For three months in the summer (12 weeks), Job A pays $325 per month with a monthly raise of $100. Job B pays $50 per week with a weekly raise of $10. Ask students which is the better paying job and why.

(RF9.5, RF9.8)

- Ask students to show that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.

(RF9.5, RF9.8)

Resources/Notes

Authorized Resource

Pre-Calculus 11

1.2 Arithmetic Series

SB: pp. 22-31
TR: pp. 19-25
BLM: 1-3, 1-5

Relations and Functions

Outcomes

Students will be expected to

RF10 Analyze geometric sequences and series to solve problems.

[PS, R]

Achievement Indicators:

RF10.1 *Identify assumptions made when identifying a geometric sequence or series.*

RF10.2 *Provide and justify an example of a geometric sequence.*

RF10.3 *Determine a rule for finding the general term of a geometric sequence.*

RF10.4 *Determine t_1, r, n or t_n in a problem that involves a geometric sequence.*

Elaborations—Strategies for Learning and Teaching

Students will continue to explore patterns with a focus shifting to geometric sequences and series. They will explore examples of geometric sequences having a "common ratio" pattern resulting in a specific formula. Students will then differentiate between a geometric sequence and a geometric series.

Similar to an arithmetic sequence, students will continue to use the same notation when working with the terms of a geometric sequence. The symbol r will represent the common ratio.

Students should provide their own examples of geometric sequences and illustrate the concept of a common ratio by determining the quotient of pairs of consecutive terms (i.e., $\frac{t_{n+1}}{t_n}$), demonstrating why any such sequence is geometric.

It is important for students to think about what makes a sequence geometric as opposed to arithmetic. They should differentiate between a common difference and a common ratio. Students should be guided through the development of the formula for a geometric sequence. Ask students to consider a sequence such as 1, 3, 9, 27, 81,... .Use the following questions to guide them through the process:

- What is the common ratio?
- Can this sequence be rewritten to show the pattern of the first term and the common ratio?
- Can you predict the formula of a geometric sequence based on the pattern 1, $1(3)^1$, $1(3)^2$, $1(3)^3$, $1(3)^4$.....?
- Can you write the pattern in general terms for any first term and common ratio ?

Students should notice that the terms of the geometric sequence with first term t_1 and a common ratio r can be written in general as $t_1 = t_1$, $t_2 = t_1(r)^1$, $t_3 = t_1(r)^2$, $t_4 = t_1(r)^3$... They should recognize the formula for an geometric sequence is $t_n = t_1(r^{n-1})$.

Ask students to draw a sketch of the geometric sequence where the x-axis represents the number of the term and the y-axis represents the term. They should notice the terms of the geometric sequence are not linear. Exposure to the graph of an exponential function may be warranted here since the graph of a geometric sequence is an exponential function with a domain belonging to the natural numbers. Students will study exponential functions and their graphs in Mathematics 3200 and 3201. Provide numerical and algebraic examples for students to evaluate and/or manipulate a given geometric sequence to determine the first term, the common ratio, or the general rule. Since students are not expected to solve exponential equations until Mathematics 3200 or 3201, questions involving determining the number of terms should be limited to examples which can be solved using simple like bases.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Performance

- Provide students with pieces of square paper of varying side lengths, *n*. Ask students to fold the paper in half repeatedly. After each fold, the area of the top surface is measured and recorded. Ask them what type of sequence is formed and to explain their reasoning.

(RF10.2)

Paper and Pencil

- Ask students to determine the first three terms of each of the following sequences: $t_n = 3^n$ and $t_n = 3(3^{n-1})$. Ask students what they notice about the value of t_1, t_2 and t_3 and the type of sequences created. They should explain their reasoning.

(RF10.1, RF10.2)

- The formula for a geometric sequence is given by $t_n = t_1(r^{n-1})$. Ask students to create examples of various geometric sequences in this form and explain what happens if $n = 1$. Ask them what type of a sequence they will continually produce if $n = 1$.

(RF10.4)

- A classmate is having trouble with the formula $t_n = t_1(r^{n-1})$. He thinks the formula should be $t_n = t_1(r^n)$. Ask students how they would explain to him that he should use r^{n-1} rather than r^n in the formula.

(RF10.3)

- You are hired to complete a job for a month that offers two different payment options. In plan A, the payment begins with $3 on day 1, $6 on day 2, $9 on day 3, etc. In plan B, the payment begins with $0.01 on day 1, $0.02 on day 2, $0.04 on day 3, etc. Ask students to determine which payment plan would be more feasible. They should explain why an employer might or might not offer a payment such as plan B.

(RF10.4)

Resources/Notes

Authorized Resource

Pre-Calculus 11

1.3 Geometric Sequences

SB: pp. 32-45
TR: pp. 26-33
BLM: 1-3, 1-6

Relations and Functions

Outcomes

Students will be expected to

RF10 Continued ...

Achievement Indicators:

RF10.5 *Determine a rule for finding the sum of n terms of a geometric series.*

RF10.6 *Determine t_1, r, n or S_n in a problem that involves a geometric series.*

RF10.7 *Solve a problem that involves a geometric sequence or series.*

Elaborations—Strategies for Learning and Teaching

Students will first work with finite geometric series. Students should be guided through the derivation of a formula for the sum of the first *n* terms for a geometric series, $S_n = \frac{t_1(r^n - 1)}{r - 1}$. Encourage students to test the formula using the following numerical example. This would give them an opportunity to review geometric sequences while being introduced to geometric series. Consider the example:

- A student is constructing a family tree. She is hoping to trace back through 10 generations to calculate the total number of ancestors he has. Determine the total number of ancestors after the 10th generation.

Students should recognize that biologically, every person has 2 parents, 4 grandparents, 8 great-grandparents, and so on. Therefore, the number of ancestors through ten generations is 2 + 4 + 8+ 16 + 32 + 64 + 128 + 256 + 512 + 1024. Let S represent the sum of this series. Ask students to multiply it by the common ratio, 2.

$$
\begin{array}{rl}
S_{10} = & 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024 \\
2S_{10} = & \quad\ \ 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024 + 2048 \\
\hline
2S_{10} - 1S_{10} = & -2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + 2048
\end{array}
$$

Students calculate the sum of the first ten terms of the series to be 2046. Going back through the generations, each person has 2046 ancestors.

Students will evaluate and/or manipulate the given geometric series to determine the first term, the common ratio, the number of terms in the sequence, the n^{th} term or the finite sum. Questions to determine the number of terms should be limited to those which can be solved using simple like terms.

It is also important to solve contextual problems that involve a geometric series. Consider a ball that is dropped from a height of 38.28 m and bounces back up 60% of the original height. Ask students to find the total distance travelled by the ball by the time it hits the ground for the tenth time.

When working through the problems, remind students of the following guidelines:

- Draw a diagram if necessary.
- Write out the terms of the sequence.
- Determine if the problem is a sequence or series.
- Determine if the problem is arithmetic or geometric.
- Construct the formula using the given information in the problem.
- Solve the problem.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

Interview

- Ask students to respond to the following questions:
  - (i) Explain why there can be no infinite geometric series with a first term of 12 and a sum of 5.
  - (ii) Explain why the sum of an infinite geometric series is positive if and only if the first term is positive.

(RF10.7)

Observation

- Invite students to play WODB (Which One Doesn't Belong?). Show students the following sequences. Each sequence could be the one which doesn't belong, but for a different reason. Observe students' reasoning for misconceptions.

$2, 1\frac{3}{5}, \frac{32}{25}, 1\frac{3}{125}, \frac{512}{625}, \ldots$	$4, 10, 16, 22, 28, \ldots$
$2, \frac{1}{3}, -1\frac{1}{3}, -3, -\frac{14}{3}, \ldots$	$2, 5, 8, 11, 14$

(RF9.5, RF10.7)

Resources/Notes

Authorized Resource

Pre-Calculus 11

1.4 Geometric Series

SB: pp. 46-57
TR: pp. 34-40
BLM: 1-3, 1-7

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/seqser.html

- WODB sequences

Relations and Functions

Outcomes

Students will be expected to

RF10 Continued ...

Achievement Indicators:

RF10.8 *Determine if a given a geometric series is convergent or divergent.*

RF10.9 *Generalize, using inductive reasoning, a rule for determining the sum of an infinite geometric series.*

Elaborations—Strategies for Learning and Teaching

To find the sum of an infinite geometric series, $S_n = \frac{t_1}{1-r}$ where $r \neq 1$, students will first need to differentiate between a convergent and divergent series. To develop the idea of convergence or divergence, expose students to a variety of geometric sequences where the common ratio is different for each. Ask students to determine the partial sum by adding the first two terms, the first three terms and so on. They will check to see if the partial sum approaches a particular value as the number of terms get larger.

Geometric Series	Partial Sum	Convergent/ Divergent
2 + 4 + 8 +16 +...	$S_1 = 2$, $S_2 = 6$, $S_3 = 14$	diverges to infinity
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+....$	$S_1 = \frac{1}{2}$, $S_2 = \frac{3}{4}$, $S_3 = \frac{7}{8}$	converges to 1
-1 + 1+ -1 + 1+	$S_1 = -1$, $S_2 = 0$, $S_3 = -1$ $S_4 = 0$	diverges
1 + 1 + 1 + 1 +...	$S_1 = 1$, $S_2 = 2$, $S_3 = 3$, $S_4 = 4$	diverges to infinity

Once students determine the common ratio for each geometric series, they should state their conclusions about convergence or divergence based on the value of r (i.e., $r > 1$,$-1 < r < 1$, $r < 1$, $r = -1$, $r = 1$). Ask students why the value of r cannot equal 0.

Students could use a concrete representation to help with their understanding of an infinite geometric series. Using a string, 1 metre long, students will cut it in half and place one of the halves stretched out on a table. Using the remaining half, they will then cut it in half so that they have two quarters. Place one of the quarters at the end of the $\frac{1}{2}$ string on the table. They now have $\frac{3}{4}$ of the string on the table. Halve the remaining quarter string so they have two eighths and place one of the eighths at the end of the string on the table. They now have $\frac{7}{8}$ of the string on the table, and so on. The goal is for students to recognize that this infinite series represents a sequence of partial sums. Students can verify, using the formula $S_n = \frac{\frac{1}{2}}{1-\frac{1}{2}}$, that the sum should equal 1.

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Suggested Assessment Strategies

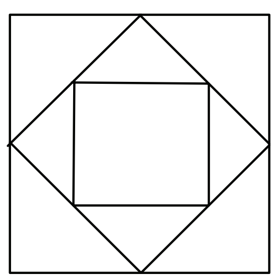
Paper and Pencil

- Ask students to determine whether the following sequences converge or diverge. They should explain their reasoning.

  (i) 8, 4, 2, 1, 0.5, ...
  (ii) $3, \frac{7}{3}, \frac{5}{3}, 1, \frac{1}{3}$
  (iii) $5^{-3}, 5^{-2}, 5^{-1}, 5^{0}, \ldots$
  (iv) $t_1 + d, t_1 + 2d, t_1 + 3d, t_1 + 4d, \ldots$

(RF10.8)

- The midpoints of a square with sides 1 m long are joined to form another square. Then the midpoints of the sides of the second square are joined to form a third square. This process is continued indefinitely to form an infinite set of smaller and smaller squares converging on the center of the original square. Determine the total length of the segments forming the sides of all the squares.



(RF10.9)

Resources/Notes

Authorized Resource

Pre-Calculus 11

1.5 Infinite Geometric Series

SB: pp. 58-65
TR: pp. 41-46
BLM: 1-3, 1-8

Suggested Resource

Resource Link:
www.k12pl.nl.ca/curr/10-12/math/2200/links/seqser.html

- square diagram

Appendix:

Outcomes with Achievement Indicators Organized by Topic (With Curriculum Guide Page References)

Topic: Trigonometry	**General Outcome**: Develop trigonometric reasoning.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
T1. Demonstrate an understanding of angles in standard position [0° to 360°]. [CN, ME, R, V]	T1.1 Sketch an angle in standard position, given the measure of the angle.	p. 22
	T1.2 Determine the quadrant in which a given angle in standard position terminates.	p. 22
	T1.3 Determine the reference angle for an angle in standard position.	pp. 22 - 24
	T1.4 Explain, using examples, how to determine the angles from 0° to 360° that have the same reference angle as a given angle.	pp. 22 - 24
	T1.5 Illustrate, using examples, that any angle from 90° to 360° is the reflection in the x-axis and/or the y-axis of its reference angle.	pp. 22 - 24
	T1.6 Draw an angle in standard position given any point P(x, y) on the terminal arm of the angle.	p. 24
	T1.7 Illustrate, using examples, that the points P(x, y), P(-x, y), P(-x, -y)and P(x, -y) are points on the terminal arms of angles in standard position that have the same reference angle.	p. 24
T2. Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position. [C, ME, PS, R, T, V]	T2.1 Determine, using the Pythagorean theorem, the distance from the origin to a point *P(x, y)* on the terminal arm of an angle.	p. 26
	T2.2 Determine the value of sin θ, cos θ, or tan θ given any point *P(x, y)* on the terminal arm of angle θ .	p. 26
	T2.3 Determine the sign of a given trigonometric ratio for a given angle, without the use of technology, and explain.	p. 26
	T2.4 Sketch a diagram to represent a problem.	pp. 26 - 34
	T2.5 Determine, without the use of technology, the value of sin θ, cos θ, or tan θ given any point *P(x, y)* on the terminal arm of angle θ, where θ = 0 °, 90 °,180 °, 270 °or 360 °.	p. 28
	T2.6 Solve, for all values of θ,an equation of the form sin $\theta = a$ or cos $\theta = a$, where $-1 \le a \le 1$, and an equation of the form tan $\theta = a$, where a is a real number.	p. 30
	T2.7 Determine the exact value of the sine, cosine or tangent of a given angle with a reference angle of 30°, 45°or 60°.	pp. 30 - 32
	T2.8 Describe patterns in and among the values of the sine, cosine and tangent ratios for angles from to 0° to 360°.	p. 34
	T2.9 Solve a contextual problem, using trigonometric ratios.	p. 34

Topic: Trigonometry	**General Outcome**: Develop trigonometric reasoning.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
T3. Solve problems, using the cosine law and sine law, including the ambiguous case. [C, CN, PS, R, T]	T3.1 Sketch a diagram to represent a problem that involves a triangle without a right angle.	p. 36
	T3.2 Solve, using primary trigonometric ratios, a triangle that is not a right triangle.	p. 36
	T3.3 Explain the steps in a given proof of the sine law and cosine law.	pp. 38, 44
	T3.4 Sketch a diagram and solve a problem, using the sine law.	p. 40
	T3.5 Describe and explain situations in which a problem may have no solution, one solution or two solutions.	pp. 42 - 44
	T3.6 Sketch a diagram and solve a problem, using the cosine law.	p. 46

<table>
<tr><td>Topic: Relations and Functions</td><td colspan="2">General Outcome: Develop algebraic and graphical reasoning through the study of relations.</td></tr>
<tr><td>Specific Outcomes
It is expected that students will:</td><td>Achievement Indicators
The following sets of indicators help determine whether students have met the corresponding specific outcome.</td><td>Page Reference</td></tr>
<tr><td>RF1. Factor polynomial expressions of the form:
• $ax^2 + bx + c, a \neq 0$
• $a^2x^2 - b^2y^2, a \neq 0, b \neq 0$
• $a(f(x))^2 + b(f(x)) + c, a \neq 0$
• $a^2(f(x))^2 - b^2(g(y))^2, a \neq 0, b \neq 0$
where a, b and c are rational numbers.
[CN, ME, R]</td><td>RF1.1 Factor a given polynomial expression that requires the identification of common factors.

RF1.2 Factor a given polynomial expression of the form:
• $ax^2 + bx + c, a \neq 0$
• $a^2x^2 - b^2y^2, a \neq 0, b \neq 0$

RF1.3 Determine whether a given binomial is a factor for a given polynomial expression, and explain why or why not.

RF1.4 Factor a given polynomial expression that has a quadratic pattern, including:
• $a(f(x))^2+b(f(x))+c, a \neq 0$
• $a^2(f(x))^2-b^2(g(y))^2, a \neq 0, b \neq 0$</td><td>p. 80

p. 80

p. 80

pp. 80 - 82</td></tr>
<tr><td>RF2. Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems.
[C, PS, R, T, V]</td><td>RF2.1 Create a table of values for $y = |f(x)|$, given a table of values for $y = f(x)$.

RF2.2 Sketch the graph of $y = |f(x)|$; state the intercepts, domain and range; and explain the strategy used.

RF2.3 Generalize a rule for writing absolute value functions in piecewise notation.

RF2.4 Solve an absolute value equation graphically, with or without technology.

RF2.5 Solve, algebraically, an equation with a single absolute value, and verify the solution.

RF2.6 Explain why the absolute value equation $|f(x)| < 0$ has no solution.

RF2.7 Determine and correct errors in a solution to an absolute value equation.

RF2.8 Solve a problem that involves an absolute value function.</td><td>pp. 152 - 154

pp. 152 - 154

pp. 154 - 156

p. 158

p. 160

p. 160

p. 160

p. 160</td></tr>
</table>

<table>
<tr><td>Topic: Relations and Functions</td><td colspan="2">General Outcome: Develop algebraic and graphical reasoning through the study of relations.</td></tr>
<tr><td>Specific Outcomes
It is expected that students will:</td><td>Achievement Indicators
The following sets of indicators help determine whether students have met the corresponding specific outcome.</td><td>Page Reference</td></tr>
<tr><td>RF3. Analyze quadratics of the form $y = a(x - p)^2 + q$, $a \neq 0$, and determine the:
• vertex
• domain and range
• direction of opening
• axis of symmetry
• x- and y-intercepts.
[CN, R,T,V]</td><td>RF3.1 Explain why a function given in the form $y = a(x - p)^2 + q$ is a quadratic function.

RF3.2 Compare the graphs of a set of functions of the form $y = ax^2$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of a.

RF3.3 Compare the graphs of a set of functions of the form $y = (x - p)^2$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of p.

RF3.4 Compare the graphs of a set of functions of the form $y = x^2 + q$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of q.

RF3.5 Determine the coordinates of the vertex for a quadratic function of the form $y = a(x - p)^2 + q$, and verify with or without technology.

RF3.6 Generalize, using inductive reasoning, a rule for determining the coordinates of the vertex for quadratic functions of the form $y = a(x - p)^2 + q$.

RF3.7 Sketch the graph of $y = a(x - p)^2 + q$, using transformations, and identify the vertex, domain and range, direction of opening, axis of symmetry and x- and y-intercepts.

RF3.8 Explain, using examples, how the values of a and q may be used to determine whether a quadratic function has zero, one or two x-intercepts.

RF3.9 Write a quadratic function in the form $y = a(x - p)^2 + q$ for a given graph or a set of characteristics of a graph.</td><td>p. 52

p. 54

p. 54

p. 54

p. 56

p. 56

p. 58

p. 60

p. 60</td></tr>
</table>

Topic: Relations and Functions	**General Outcome**: Develop algebraic and graphical reasoning through the study of relations.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
RF4. Analyze quadratic functions of the form $y = ax^2 + bx + c$, $a \neq 0$, to identify characteristics of the corresponding graph, including: • vertex • domain and range • direction of opening • axis of symmetry • x-and y-intercepts and to solve problems. [CN, PS, R, T, V]	RF4.1 Determine the characteristics of a quadratic function given in the form $y = ax^2 + bx + c$, and explain the strategy used.	p. 62
	RF4.2 Sketch the graph of a quadratic function given in the form $y = ax^2 + bx + c$.	pp. 62 - 64
	RF4.3 Explain the reasoning for the process of completing the square as shown in a given example.	pp. 64 - 66
	RF4.4 Write a quadratic function given in the form $y = ax^2 + bx + c$ as a quadratic function in the form $y = a(x - p)^2 + q$ by completing the square.	pp. 66 - 68
	RF4.5 Identify, explain and correct errors in an example of completing the square.	p. 70
	RF4.6 Verify, with or without technology, that a quadratic function in the form $y = ax^2 + bx + c$ represents the same function as a given quadratic function in the form $y = a(x - p)^2 + q$.	p. 70
	RF4.7 Write a quadratic function that models a given situation, and explain any assumptions made.	p. 72
	RF4.8 Solve a problem, with or without technology, by analyzing a quadratic function.	p. 72

Topic: Relations and Functions	**General Outcome**: Develop algebraic and graphical reasoning through the study of relations.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
RF5. Solve problems that involve quadratic equations. [C, CN, PS, R, T, V]	RF5.1 Explain, using examples, the relationship among the roots of a quadratic equation, the zeros of the corresponding quadratic function and the *x*-intercepts of the graph of the quadratic function.	p. 78
	RF5.2 Solve a quadratic equation of the form $ax^2 + bx + c = 0$ by using strategies such as: • determining square roots • factoring • completing the square • applying the quadratic formula • graphing its corresponding function.	pp. 78, 82 - 84
	RF5.3 Derive the quadratic formula, using deductive reasoning.	p. 84
	RF5.4 Identify and correct errors in a solution to a quadratic equation.	p. 86
	RF5.5 Select a method for solving a quadratic equation, justify the choice, and verify the solution.	p. 86
	RF5.6 Explain, using examples, how the discriminant may be used to determine whether a quadratic equation has two, one, or no real (i.e. imaginary) roots; and relate the number of zeros to the graph of the corresponding quadratic function.	pp. 86 - 88
	RF5.7 Solve a problem by: • analyzing a quadratic equation • determining and analyzing a quadratic equation.	p. 90

Topic: Relations and Functions	**General Outcome**: Develop algebraic and graphical reasoning through the study of relations.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
RF6. Solve, algebraically and graphically, problems that involve systems of linear-quadratic and quadratic-quadratic equations in two variables. [CN, PS, R, T, V]	RF6.1 Explain the meaning of the points of intersection of a system of linear-quadratic or quadratic-quadratic equations.	p. 174
	RF6.2 Explain, using examples, why a system of linear-quadratic or quadratic-quadratic equations may have zero, one, two or an infinite number of solutions.	p. 174
	RF6.3 Determine and verify the solution(s) of a system of linear-quadratic or quadratic-quadratic equations graphically, with and without technology.	p. 176
	RF6.4 Determine and verify the real solution(s) of a system of linear-quadratic or quadratic-quadratic equations algebraically.	pp. 176-178
	RF6.5 Model a situation, using a system of linear-quadratic or quadratic-quadratic equations.	p. 178
	RF6.6 Relate a system of linear-quadratic or quadratic-quadratic equations to the context of a given problem.	p. 178
	RF6.7 Solve a problem that involves a system of linear-quadratic or quadratic-quadratic equations, and explain the strategy used.	p. 180

Topic: Relations and Functions	**General Outcome**: Develop algebraic and graphical reasoning through the study of relations.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
RF7. Demonstrate an understanding of the product, quotient and power laws of logarithms. [C, CN, ME, R, T]	RF7.1 Explain, using examples, how test points can be used to determine the solution region that satisfies a linear inequality.	pp. 186-188
	RF7.2 Explain, using examples, when a solid or broken line should be used in the solution for a linear inequality.	pp. 186-188
	RF7.3 Sketch, with or without technology, the graph of a linear inequality.	pp. 186-188
	RF7.4 Solve a problem that involves a linear inequality.	p. 188
	RF7.5 Explain, using examples, how test points can be used to determine the solution region that satisfies a quadratic inequality.	p. 192
	RF7.6 Explain, using examples, when a solid or broken line should be used in the solution for a quadratic inequality.	p. 192
	RF7.7 Sketch, with or without technology, the graph of a quadratic inequality.	p. 192
	RF7.8 Solve a problem that involves a quadratic inequality.	p. 192
RF8. Solve problems that involve quadratic inequalities in one variable. [CN, PS, V]	RF8.1 Determine the solution of a quadratic inequality in one variable, using strategies such as case analysis, graphing, roots and test points, or sign analysis; and explain the strategy used.	p. 190
	RF8.2 Represent and solve a problem that involves a quadratic inequality in one variable.	p. 190
	RF8.3 Interpret the solution to a problem that involves a quadratic inequality in one variable.	p. 190

Topic: Relations and Functions	**General Outcome**: Develop algebraic and graphical reasoning through the study of relations.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
RF9. Analyze arithmetic sequences and series to solve problems. [CN, PS, R]	RF9.1 Identify the assumption(s) made when defining an arithmetic sequence or series.	p. 198
	RF9.2 Provide and justify an example of an arithmetic sequence.	pp. 198 - 200
	RF9.3 Determine a rule for finding the general term of an arithmetic sequence.	pp. 198 - 200
	RF9.4 Determine t_1, d, n, or t_n in a problem that involves an arithmetic sequence.	p. 200
	RF9.5 Solve a problem that involves an arithmetic sequence or series.	pp. 200, 206
	RF9.6 Describe the relationship between arithmetic sequences and linear functions.	pp. 200 - 202
	RF9.7 Determine a rule for finding the sum of n terms of an arithmetic series.	pp. 202 - 204
	RF9.8 Determine t_1, d, n, or S_n in a problem that involves an arithmetic series.	p. 206
RF10. Analyze geometric sequences and series to solve problems. [PS, R]	RF10.1 Identify assumptions made when identifying a geometric sequence or series.	p. 208
	RF10.2 Provide and justify an example of a geometric sequence.	p. 208
	RF10.3 Determine a rule for finding the general term of a geometric sequence.	p. 208
	RF10.4 Determine t_1, r, n, or t_n in a problem that involves a geometric sequence.	p. 208
	RF10.5 Determine a rule for finding the sum of n terms of a geometric series.	p. 210
	RF10.6 Determine t_1, r, n, or S_n in a problem that involves a geometric series.	p. 210
	RF10.7 Solve a problem that involves a geometric sequence or series.	p. 210
	RF10.8 Determine if a given a geometric series is convergent or divergent.	p. 212
	RF10.9 Generalize, using inductive reasoning, a rule for determining the sum of an infinite geometric series.	p. 212

Topic: Relations and Functions	**General Outcome**: Develop algebraic and graphical reasoning through the study of relations.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
RF11. Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). [CN, R, T, V]	RF11.1 Compare the graph of $y=\frac{1}{f(x)}$ to the graph of $y = f(x)$.	pp. 162 - 168
	RF11.2 Identify, given a function $f(x)$, values of x for which $y=\frac{1}{f(x)}$ will have vertical asymptotes; and describe their relationship to the non-permissible values of the related rational expression.	pp. 162 - 168
	RF11.3 Graph, with or without technology, $y=\frac{1}{f(x)}$, given $y = f(x)$ as a function or a graph, and explain the strategies used.	pp. 162 - 168
	RF11.4 Graph, with or without technology, $y= f(x)$ given $y=\frac{1}{f(x)}$ as a function or a graph, and explain the strategies used.	p. 168

Topic: Algebra and Number	**General Outcome**: Develop algebraic reasoning and number sense.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
AN1. Demonstrate an understanding of the absolute value of real numbers. [R, V]	AN1.1 Determine the distance of two real numbers of the form $\pm a$, $a \in \mathbb{R}$, from 0 on a number line, and relate this to the absolute value of a ($\lvert a \rvert$).	p. 150
	AN1.2 Determine the absolute value of a positive or negative real number.	p. 150
	AN1.3 Explain, using examples, how distance between two points on a number line can be expressed in terms of absolute value.	p. 150
	AN1.4 Determine the absolute value of a numerical expression.	p. 150
	AN1.5 Compare and order the absolute values of real numbers in a given set.	p. 150
AN2. Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands. [CN, ME, PS, R]	AN2.1 Compare and order radical expressions with numerical radicands in a given set.	p. 96
	AN2.2 Express an entire radical with a numerical radicand as a mixed radical.	p. 96
	AN2.3 Express a mixed radical with a numerical radicand as an entire radical.	p. 96
	AN2.4 Explain, using examples, that $(-x)^2 = x^2$, $\sqrt{x^2} = \lvert x \rvert$ and $\sqrt{x^2} \neq \pm x$; e.g., $\sqrt{9} \neq \pm 3$.	p. 98
	AN2.5 Identify the values of the variable for which a given radical expression is defined.	p. 100
	AN2.6 Express an entire radical with a variable radicand as a mixed radical.	p. 100
	AN2.7 Express a mixed radical with a variable radicand as a entire radical.	p. 100
	AN2.8 Perform one or more operations to simplify radical expressions with numerical or variable radicands.	pp. 102 - 106
	AN2.9 Rationalize the denominator of a rational expression with monomial or binomial denominators.	pp. 108 - 110
	AN2.10 Describe the relationship between rationalizing a binomial denominator of a rational expression and the product of the factors of a difference of squares expression.	p. 110

Topic: Algebra and Number	**General Outcome**: Develop algebraic reasoning and number sense.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
AN3. Solve problems that involve radical equations (limited to square roots with non-negative radicands). [C, PS, R]	AN3.1 Determine any restrictions on values for the variable in a radical equation.	pp. 112 - 118
	AN3.2 Determine the roots of a radical equation algebraically, and explain the process used to solve the equation.	pp. 112 - 118
	AN3.3 Verify, by substitution, that the values determined in solving a radical equation algebraically are roots of the equation.	pp. 112 - 118
	AN3.4 Explain why some roots determined in solving a radical equation algebraically are extraneous	pp. 112 - 118
	AN3.5 Solve problems by modeling a situation using a radical equation.	p. 118
AN4. Determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials). [C, ME, R]	AN4.1 Explain why a given value is non-permissible for a given rational expression.	pp. 124 - 128
	AN4.2 Determine the non-permissible values for a rational expression.	pp. 124 - 128
	AN4.3 Compare the strategies for writing equivalent forms of rational expressions to the strategies for writing equivalent forms of rational numbers.	pp. 128 - 130
	AN4.4 Determine a rational expression that is equivalent to a given rational expression by multiplying the numerator and denominator by the same factor (limited to a monomial or a binomial), and state the non-permissible values of the equivalent rational expression.	pp. 128 - 130
	AN4.5 Simplify a rational expression.	p. 130
	AN4.6 Explain why the non-permissible values of a given rational expression and its simplified form are the same.	p. 130
	AN4.7 Identify and correct errors in a given simplification of a rational expression, and explain the reasoning.	p. 132

Topic: Algebra and Number	**General Outcome**: Develop algebraic reasoning and number sense.	
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following sets of indicators help determine whether students have met the corresponding specific outcome.*	**Page Reference**
AN5. Perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials). [CN, ME, R]	AN5.1 Compare the strategies for performing a given operation on rational expressions to the strategies for performing the same operation on rational numbers.	p. 134
	AN5.2 Determine the non-permissible values when performing operations on rational expressions.	pp. 134 - 136
	AN5.3 Determine, in simplified form, the product or quotient of rational expressions.	p. 134
	AN5.4 Determine, in simplified form, the sum or difference of rational expressions with the same denominator.	p. 136
	AN5.5 Determine, in simplified form, the sum or difference of rational expressions in which the denominators are not the same and which may or may not contain common factors.	p. 136
	AN5.6 Simplify an expression that involves two or more operations on rational expressions.	p. 138
AN6. Solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials or trinomials). [C, PS, R]	AN6.1 Determine the non-permissible values for the variable in a rational equation.	pp. 140 - 142
	AN6.2 Determine the solution to a rational equation algebraically, and explain the strategy used to solve the equation.	pp. 140 - 142
	AN6.3 Explain why a value obtained in solving a rational equation may not be a solution of the equation.	pp. 140 - 142
	AN6.4 Solve problems by modeling a situation using a rational equation.	p. 144

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September 2017
ISBN: 978-1-55146-623-1